The Global Phase Plane Analysis of Three Vortex Interactions

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Abstract

We investigate global phase planes in point-vortex dynamics in a two-dimensional, inviscid, incompressible fluid. We derive a symplectic reduction of a system involving three vortices, initially employing Jacobi coordinates followed by Lie-Poisson reduction. We conduct a global phase analysis of a three-vortex problem with arbitrary circulations with novel bifurcation analysis. This reduction method eliminates coordinate singularities that made understanding the dynamics challenging.

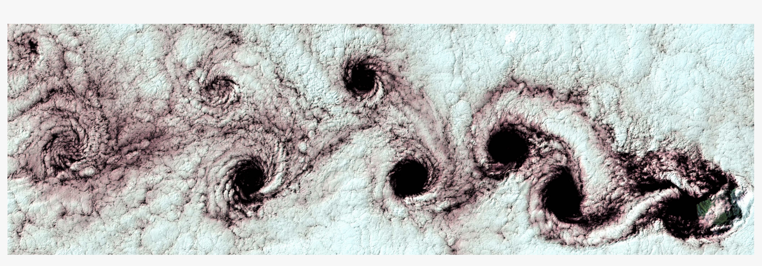
The Point Vortex Model

The N vortex positions satisfy:

$$\dot{x}_i = -\frac{\Gamma_j}{2\pi}\sum_{j\neq i}\frac{(y_i - y_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2},$$
$$\dot{y}_i = +\frac{\Gamma_j}{2\pi}\sum_{j\neq i}\frac{(x_i - x_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}.$$

with the conserved Hamiltonian (3),

$$\mathcal{H}(\mathbf{r}) = \sum_{1\leq i<j\leq N}\frac{-\Gamma_i\Gamma_j \log\left\|\mathbf{r}_i - \mathbf{r}_j\right\|^2}{4\pi}.$$



Vortices in the atmosphere.

Previous Studies and Limitations

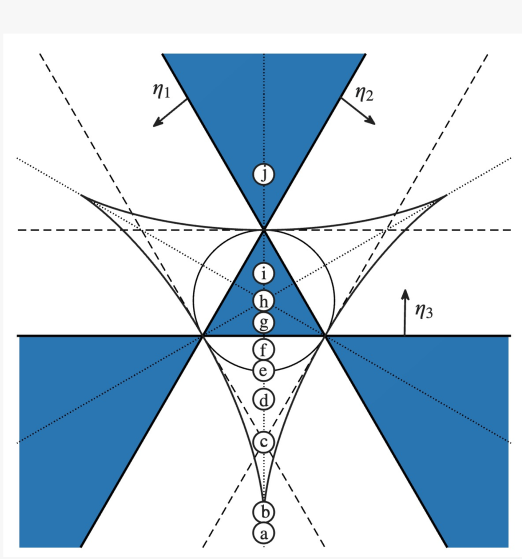
Previous studies by Gröbli (2) introduced a coordinate system based on the triangle side lengths with vertices at the three vortices. The coordinate system has the following issues:

- Singularity in equations at collinear configurations.
- Nonphysical singularities introduced during reduction.

Under the assumption

$$\eta_1 + \eta_2 + \eta_3 = 1.$$

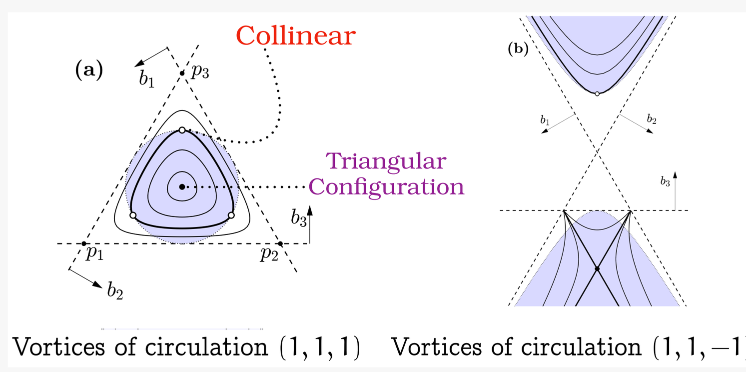
Aref derived a bifurcation diagram showing how the phase space depends on the circulations.



Aref's barycentric coordinates illustrate how three-vortex dynamics vary with the sign of κ_2: blue shading shows spherical dynamics ($\kappa_2 > 0$), while the unshaded area represents hyperbolic dynamics ($\kappa_2 < 0$) in the XY plane projection.

Aref's Phase Planes

- Portions lying outside the shaded regions lack physical meaning.
- Dynamics singular at collinear relative equilibria (∘).
- Aref's phase space based on his method is hard to follow.



Phase diagrams in trilinear coordinates for vortices of circulations $(1, 1, -1)$ in Aref's trilinear coordinates.

I. Jacobi Coordinates

- The Jacobi coordinate transformation is used to simplify the formulation in n-body problems.
- It replaces the coordinates of two vortices at positions z_j and z_{j+1} by their displacement $Z_j = z_{j+1} - z_j$ and their center of vorticity. The process is applied iteratively.

Jacobi coordinates Z for three vortices with corresponding *reduced circulations* κ are defined as:

$$Z_1 = z_1 - z_2; \quad Z_2 = \frac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2} - z_3;$$
$$Z_3 = \frac{\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3}{\Gamma_1 + \Gamma_2 + \Gamma_3};$$
$$\kappa_1 = \frac{\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}; \quad \kappa_2 = \frac{(\Gamma_1 + \Gamma_2)\Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3};$$
$$\kappa_3 = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where z_i are the vortex positions. WLOG, the center of vorticity can be placed at the origin. We assume $\Gamma_1 \geq \Gamma_2 > 0$.

II. Lie-Poisson Reduction

We use *Lie-Poisson* reduction for reformulating dynamics: the evolution equation

$$\dot{\mu} = \mathbf{D}_{\Gamma^{-1}}\frac{\delta h}{\delta \mu}\mu - \mu\frac{\delta h}{\delta \mu}\mathbf{D}_{\Gamma^{-1}},$$
$$\mu := ZZ^* = \begin{bmatrix} |Z_1|^2 & Z_1 Z_2^* \\ Z_2 Z_1^* & |Z_2|^2 \end{bmatrix},$$

where the conserved quantity Θ is defined by:

$$\Theta^2 = Z^2 + 4\kappa_1\kappa_2\left(X^2 + Y^2\right).$$

Hamiltonian and Angular Impulse in Jacobi-Lie-Poisson

In Jacobi coordinates, the Hamiltonian H and angular impulse Θ are:

$$h = -\frac{\Gamma_1\Gamma_2}{2}\log\|Z_1\|^2 - \frac{\Gamma_2\Gamma_3}{2}\log\left\|Z_2 - \frac{\kappa_1}{\Gamma_2}Z_1\right\|^2$$
$$-\frac{\Gamma_1\Gamma_3}{2}\log\left\|Z_2 + \frac{\kappa_1}{\Gamma_1}Z_1\right\|^2;$$
$$\Theta = \kappa_1\|Z_1\|^2 + \kappa_2\|Z_2\|^2.$$

Dynamics Based on κ_2 Sign

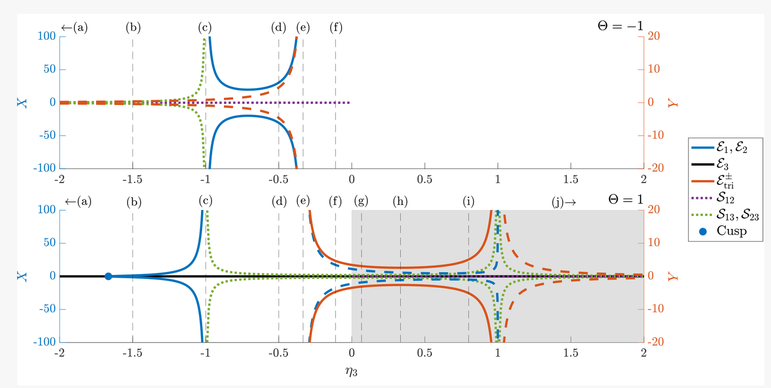
- $\kappa_2 < 0$: Represents a **two-sheeted hyperboloid** in (X, Y, Z, Θ) coordinates.
- $\kappa_2 > 0$: Represents a **sphere** in (X, Y, Z, Θ) coordinates.

Evolution Equations

The system's dynamics are given by:

$$\frac{dX}{dt} = 4h_Z Y, \quad \frac{dZ}{dt} = -4h_X Y,$$
$$\frac{dY}{dt} = 4h_Z X - \frac{h_X Z}{\kappa_1\kappa_2}.$$

Bifurcation Diagram for Two Equal Vortices

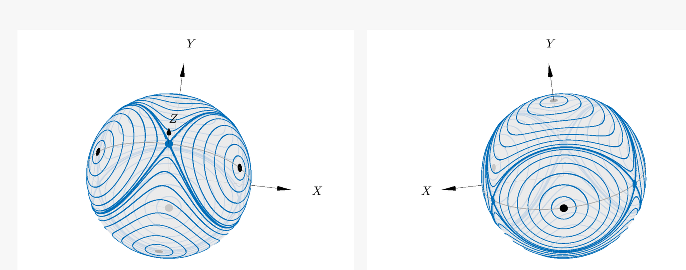


The bifurcation diagram for $\eta_1 = \eta_2$ and variable η_3 is presented, with separate plots for the $\Theta = -1$ and $\Theta = 1$ cases. The solid lines represent stable equilibria, while the dotted lines indicate unstable singularities. The points corresponding to Arefs barycentric coordinates are indicated within the diagram.

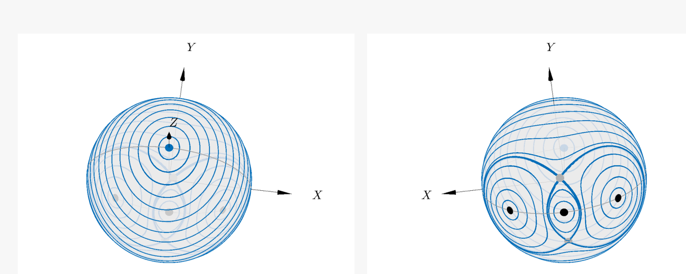
- The bifurcation at $\eta_3 = -1$ is a *pitchfork bifurcation.*
- All other bifurcations occur as fixed points escape to infinity and later reappear, occasionally transitioning between the $\Theta = \pm 1$ surfaces.

Two Equal Vortices: $\left(\frac{1-\Gamma_3}{2}, \frac{1-\Gamma_3}{2}, \Gamma_3\right)$

Two Spherical Cases



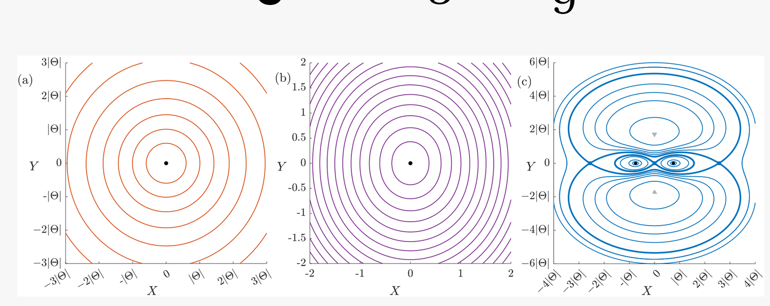
The global phase space for the case **h** with $\Gamma_3 = \frac{1}{3}$, showing "front" and "back" views of the sphere. The singularities are represented by black dots, the collinear equlibria by blue, and the triangular equilibria by gray.



The global phase space for the case **j** with $\Gamma_3 = 5$.

- From Region j to Region h: Equilibrium points change from centers to saddles when vortices move from Region 6 to Region 1.
- The equator $Y = 0$, representing collinear vortex configurations, and the meridians, corresponding to isosceles triangle formations.

Region f: $\Gamma_3 = -\frac{1}{9}$

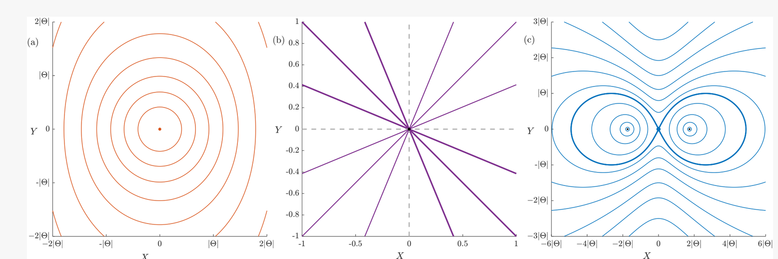


Phase planes for $\Gamma_3 = -\frac{1}{9}$.

From **Region f** to **Region e**:

- $\Theta < 0$, the nature of the singularity does not change.
- For $\Theta = 0$, the families of periodic orbits for $\Gamma_3 = -\frac{1}{9}$ collapses when $\Gamma = -\frac{1}{3}$.
- For $\Theta > 0$, the equilibria intersecting at the separatrix for $\Gamma_3 = -\frac{1}{9}$ goes off to infinity when $\Gamma_3 = -\frac{1}{3}$.

Vortex Collapse[1]: $\Gamma_3 = -\frac{1}{3}$

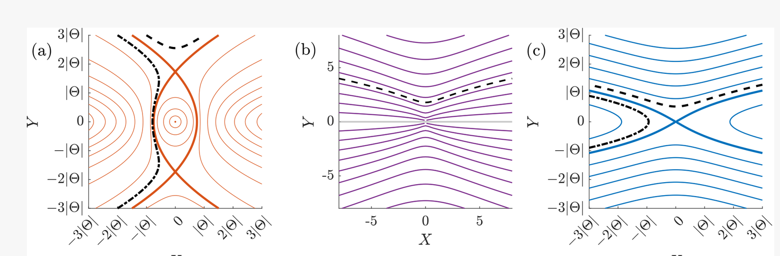


The phase planes for vortex-collapse at $\Gamma_3 = -\frac{1}{3}$.

From **Region e** to **Region c**:

- $\Theta < 0$, the nature of the singularity at the origin does not change, but new equilibria appears at the separatrix.
- For $\Theta = 0$, the line $Y = 0$ is singular from $\Gamma_3 = -\frac{1}{3}$ to $\Gamma_3 = -1$.
- For $\Theta > 0$, the equilibria at the separatrix go off to $\pm\infty$, leaving with one equilibrium (collinear state) at the origin.
- The phase planes shows the Direct scattering[2] (dash-dot) and Exchange Scattering (dash).

Vortex-Dipole Scattering Problem[3]: $\Gamma_3 = -1$



The XY phase planes of system when $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, -1)$. (a) The case $\Theta < 0$ with singularity (point) and triangular configurations at the intersections of the thick curves. (b) The case $\Theta = 0$. The gray line $Y = 0$ is singular. (c) the case $\Theta > 0$ with collinear equilibrium at the separatrix intersection (1).

Acknowledgements

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References

(1) A. Anurag et al. ``A new canonical reduction of three-vortex motion and its application to vortex-dipole scattering''. In: *Physics of Fluids* (2024).

(2) W. Gröbli. ``Spezielle probleme über die Bewegung geradliniger paralleler Wirbelfäden''. PhD thesis. Georg-August-Universität Göttingen, 1877.

(3) G. Kirchhoff. *Vorlesungen über mathematische Physik: Mechanik*. Vol. 1. Vorlesungen über mathematische Physik. Teubner, Leipzig, 1876.

[1] A famous example of the *vortex collapse* case, where $\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_1\Gamma_3 = 0$.

[2] The dipole is formed by same vortices after interacting with initial vortex 2; else Exchange.

[3] Region (c) in 8.