GENERALIZATION OF LEAPFROGGING ORBITS OF POINT VORTICES

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What is a point vortex?

- A point vortex is a solution to 2D inviscid incompressible fluid equations with vorticity confined to a single point.
- In a point vortex model, the solution is represented by three quantities, strength (circulation), position, and orientation.

What is the point vortex model?

- It is the idealization of Euler's equations derived by Helmholtz, which described the motion of two or more interacting point vortices.
- The point vortex model is a Hamiltonian system, and this allows us to use the mathematical theory and tools of Hamiltonians to study it.

Helmholtz derivation of the vortex induction equations

- Let $\mathbf{u}(\mathbf{x}, t)$ solve the 2D Euler's equation.
- Particles advected according to $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$.
- By Helmholtz decomposition

$$\mathbf{u} = \nabla\phi + \nabla \times \psi$$

where $\triangle\psi = -\omega$ and $\omega = \nabla \times \mathbf{u}$.

- Let vorticity be concentrated at N points $\mathbf{x}_i$ of circulation Γ_i:

$$\omega(\mathbf{x}) = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{x} - \mathbf{x_i}).$$

- Velocity due to each vortex given by the Green's function for 2D Poisson equation, yielding evolution equations:

$$\dot{x}_i = -\frac{1}{2\pi} \sum_{j \neq i}^{N} \Gamma_j \frac{(y_i - y_j)}{||\mathbf{x}_j - \mathbf{x}_i||^2} \quad \text{and} \quad \dot{y}_i = +\frac{1}{2\pi} \sum_{j \neq i}^{N} \Gamma_j \frac{(x_i - x_j)}{||\mathbf{x}_j - \mathbf{x}_i||^2}.$$

Kirchhoff's Hamiltonian Representation

Consider N-vortices located at $\mathbf{r}_i = (x_i, y_i)$ with the strength Γ_i, then the system of ODEs describing the N-vortex motion can be described by the Hamiltonian,

Hamiltonian

$$\mathcal{H}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N) = -\frac{1}{4\pi} \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j \log \|\mathbf{r}_i - \mathbf{r}_j\|^2.$$

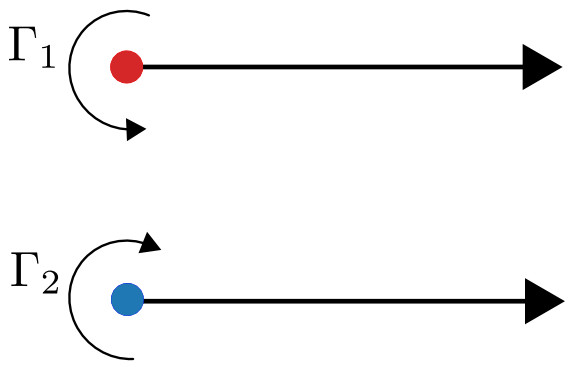
System of $2N$ point vortex equations:

$$\Gamma_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial y_i}, \qquad \Gamma_i \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial x_i}.$$

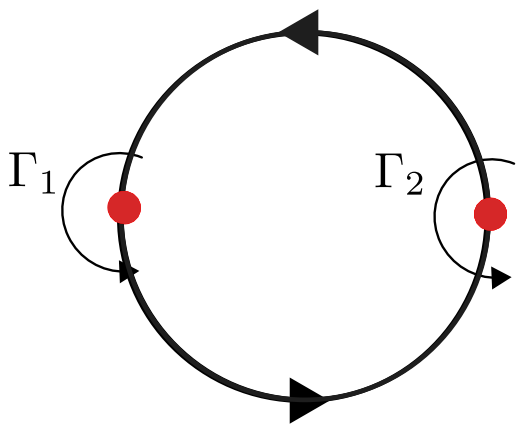
Motion of two vortices

- Opposite-signed vortices move in parallel along straight lines
- Like-signed vortices move along a circular path

(a) $\Gamma_1 = -\Gamma_2 > 0$



(b) $\Gamma_1 = \Gamma_2 > 0$



These are examples of relative equilibria, i.e. the solutions are stationary when observed in appropriate (translating or rotating) reference frame.

Leapfrogging Motion in Vortex Rings

Leapfrogging smoke rings

Figure: Credit: Robert Morton, Irvine Lab, University of Chicago

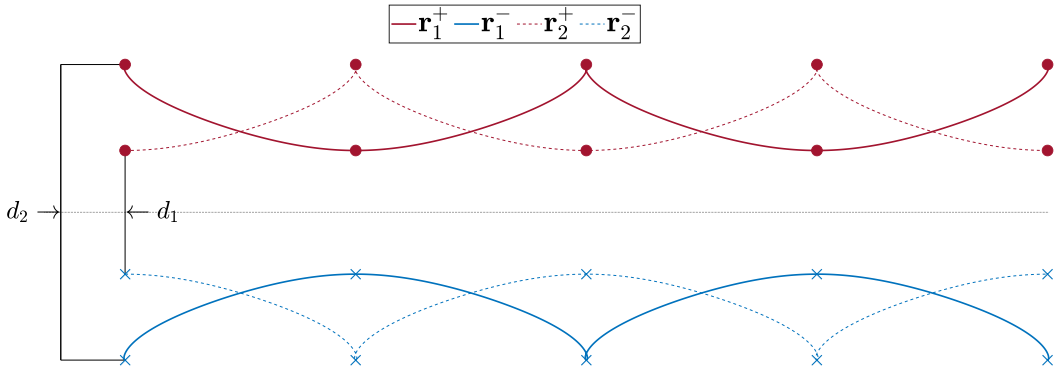
Leapfrogging in the pool

Figure: Credit: thephysicsgirl on Instagram

Leapfrogging motion in a vortex rings is the historical motivation to describe the leapfrogging motion of point vortices.
What is a leapfrogging motion of point vortices?

- A type of motion that involves two pairs of point vortices leaping past each other.
- In a leapfrogging motion, two pairs of point vortices move past each other in a way that resembles the movement of a pair of leaping frogs.

Leapfrogging Motion of Point Vortices



Love (1893) showed the existence of a relative periodic orbit that depends on a ratio $\alpha = \frac{d_1}{d_2}$, and one-parameter family of relative periodic orbits exist for $\alpha > \alpha_{\text{exist}} = 3 - 2\sqrt{2}$.

α near 1

Smaller α

Prior Results: Acheson (2000) Eur. J. Phys.

Acheson found (via direct numerical simulation) that for $\alpha \gtrsim 0.382$ the motion is stable and, further:

For $0.172 < \alpha < 0.29$, unstable leapfrogging orbits disintegrate:

For $0.29 < \alpha < 0.382$, motion goes into walkabout orbit:

walkabout

disintegration

Goodman and Behring (2022) proved that this bifurcation occurs at $\alpha = \phi^{-2} \approx 0.382$, where ϕ is the golden ratio.

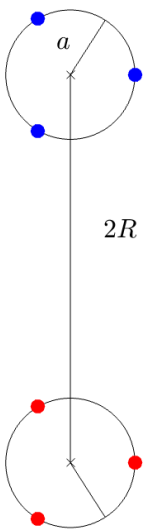
Rigidly-Rotating Configurations

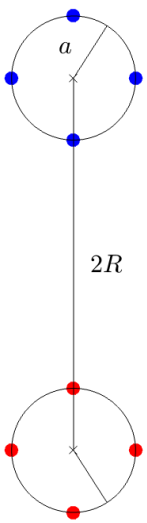
What makes it interesting to study the generalized leapfrogging problem?

- Kelvin (1872) studied rigidly-rotating configurations of N identical vortices, where he first derived the linear stability for $N > 7$ to be unstable, while $N < 7$ (linearly stable), and rings of $N = 7$ are linearly neutral but non-linearly unstable,
- Such rigidly-rotating configurations are called relative equilibria.
- Periodic solutions for which the configuration of vortices rotates rigidly about the center of vorticity play a crucial role. Such solutions are known as relative equilibria, since they are fixed points in a rotating coordinate system.

Constructing the Leapfrogging Initial Conditions

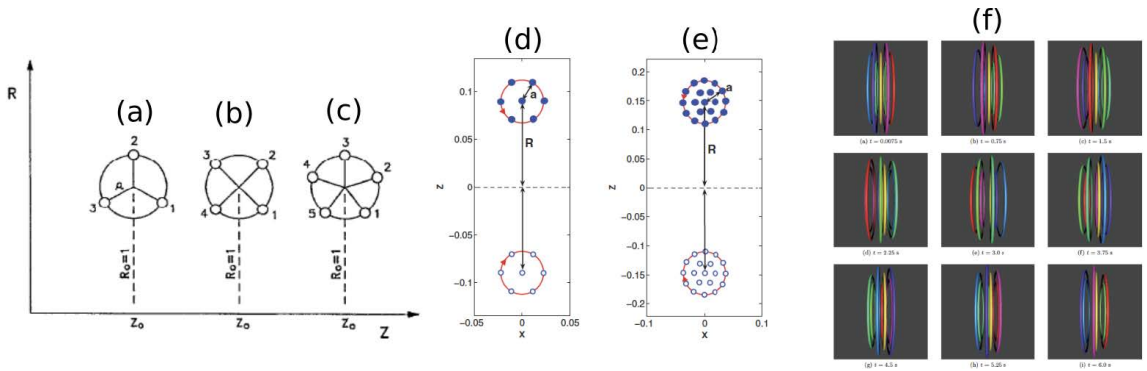
- The ring of N like signed vortices is a generalization of the two vortex case with $\Gamma_1 = \Gamma_2$.
- Therefore, placing two rings that mirror each other is the generalization of the vortex leapfrogging we seek.





Generalization of Leapfrogging: Prior Results

Previous groups have studied the initial value problem but have not constructed the relative periodic orbits.



- (**a**-**c**). The near-leapfrog initial conditions studied numerically by Konstantinov.
- (**d**-**e**). The near-leapfrog initial conditions studied numerically by Wacks et al.
- **f**. A leapfrogging-like motion of 7 vortex rings.

Generalized Leapfrogging Model

The leapfrogging motion of rigidly-rotating cluster of radius a at a distance R above the x-axis and its mirror image, consisting of vortices of opposite circulation, at a distance R below, depends on

$$\beta = \frac{a}{R}.$$

- For $\beta \ll 1$, the far flow field due to each cluster will approach that of a point vortex and we expect stable leapfrogging for $N \leq 6$.

What happens when we decrease the distance between two clusters?

- We will use the numerical continuation to construct the relative periodic orbits and analyze their linear stability.

Why the need to use numerics?

- It is generally impossible to find solutions analytically. Therefore, we need to use numerics.
- We will discuss the specific numerical methods for the continuation of periodic solutions.

Parameter Continuation

Continuation schemes are used to determine how solutions of the system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \alpha), \mathbf{x} \in \mathbb{R}^n, \alpha \in \mathbb{R}, \mathbf{F} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \tag{1}$$

vary with a certain parameter, here α.

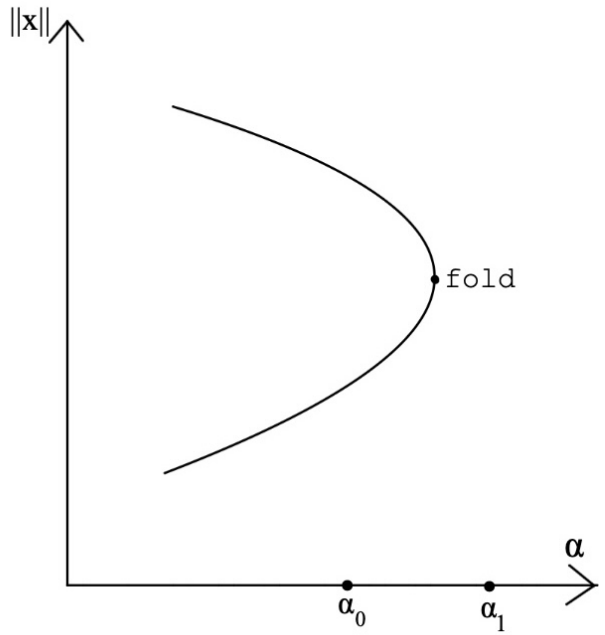
- The idea of parameter continuation is to continue a given equilibria of the system (1) by varying α.
- Given a solution, $(\mathbf{x}_0, \alpha_0)$, we aim to find $(\mathbf{x}_1, \alpha_1)$ on the same branch using Newton's method.
- The problem reduces to:

$$\mathbf{F}(\mathbf{x}_1, \alpha_1) = 0, \text{ where } \alpha_1 = \alpha_0 + \Delta\alpha.$$

Drawback of Parameter continuation

- The parameter continuation method is bound to break down at turning points, where

$$\det\{\mathbf{F}_{\mathbf{x}}\} = 0.$$



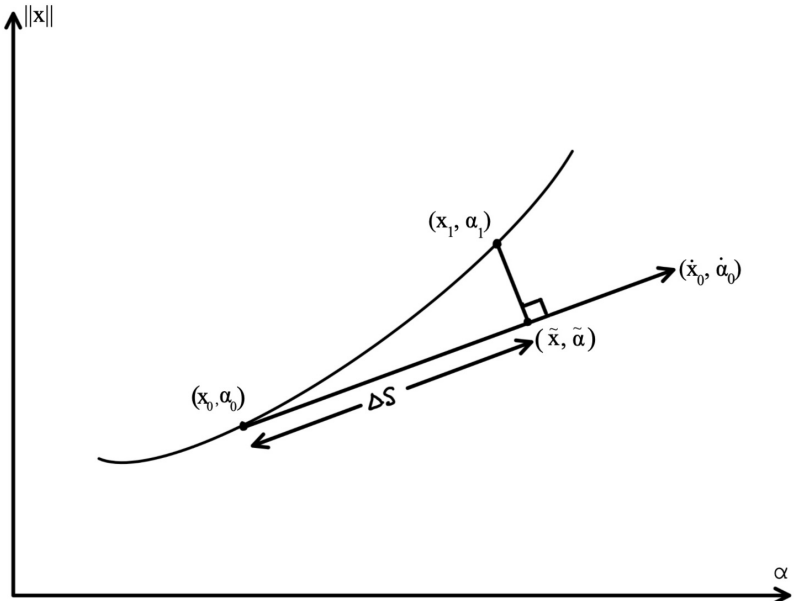
- Keller (1987) proposed a continuation scheme called **pseudo-arclength continuation** to overcome this problem.
- The Pseudo-arclength continuation scheme allows the continuation of a solution family past a fold.

Pseudo-arclength Continuation

- In this scheme, $\mathbf{x}$ and α are considered to the function of s, arc-length.
- Now, the idea is to find the roots of the system

$$\mathbf{F}(\mathbf{x}(s), \alpha(s)) = 0. \tag{2}$$

- $(\mathbf{x}_1, \alpha_1)$ satisfies the orthogonality condition below.



Pseudo-arclength Continuation

- Differentiating both sides of the Equation (2) with respect to s

$$\mathbf{F}_{\mathbf{x}}(\mathbf{x}, \alpha)\ \dot{\mathbf{x}} + \mathbf{F}_{\alpha}(\mathbf{x}, \alpha)\ \dot{\alpha} = 0,$$

  gives rise to n linear algebraic equations in $(n+1)$ unknowns $\dot{\mathbf{x}}$ and $\dot{\alpha}$.

- The normalized arc-length condition

$$\dot{\mathbf{x}}^{\mathsf{T}}\dot{\mathbf{x}} + \dot{\alpha}^2 = 1. \tag{3}$$

  is used to determine the solutions of the above system uniquely.

- The orthogonality condition is

$$\langle(\mathbf{x}_1 - \tilde{\mathbf{x}}, \alpha_1 - \tilde{\alpha})|(\mathbf{x}_0 - \tilde{\mathbf{x}}, \alpha_0 - \tilde{\alpha})\rangle = 0, \tag{4}$$

Equations (3) and (4) give us

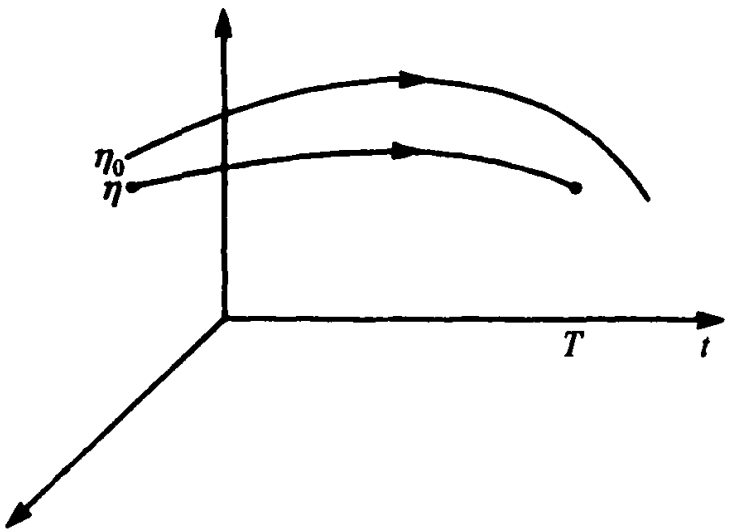
$$\mathbf{F}(\mathbf{x}_1, \alpha_1) = 0, \text{ and } (\mathbf{x}_1 - \mathbf{x}_0)^{\mathsf{T}}\dot{\mathbf{x}}_0 + (\alpha_1 - \alpha_0)\dot{\alpha}_0 - \Delta s = 0.$$

Shooting Method

We are interested in continuing the branches of relative periodic orbits, not equilibria.

- Shooting method is used to find a single periodic orbit of (2) by reformulating the system to a BVP.
- Our goal is to seek an initial condition $\mathbf{x}(0) = \boldsymbol{\eta}$ and a solution $\mathbf{x}(t;\boldsymbol{\eta})$ with a minimal period T such that

$$\mathbf{x}(T, \boldsymbol{\eta}) = \boldsymbol{\eta}.$$



- $T, \boldsymbol{\eta}$ are chosen close to $T_0, \boldsymbol{\eta}_0$ in such a way that

$$\mathbf{x}(T_0 + \delta T, \boldsymbol{\eta}_0 + \delta\boldsymbol{\eta}) - (\boldsymbol{\eta}_0 + \delta\boldsymbol{\eta}) \simeq 0. \tag{5}$$

Linearization of Equation (5)

$$\left[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}(T_0, \boldsymbol{\eta}_0) - \mathbf{I}\right]\delta\boldsymbol{\eta} + \left[\frac{\partial \mathbf{x}}{\partial T}(T_0, \boldsymbol{\eta}_0)\right]\delta T = \boldsymbol{\eta}_0 - \mathbf{x}(T_0, \boldsymbol{\eta}_0), \tag{6}$$

where $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ is an $n \times n$ matrix, $\mathbf{I}$ is an $n \times n$ identity matrix, and $\frac{\partial \mathbf{x}}{\partial T}$ is an $n \times 1$ vector.

- We need to determine the matrix $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ at $(T_0, \boldsymbol{\eta}_0)$ from (6).

- $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ is determined by linearized Equation (6).
- Once $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ at $(T_0, \boldsymbol{\eta}_0)$ is known, from Equation (6) we have a system of n equations in $n+1$ unknowns, $\delta\boldsymbol{\eta}$ and δT.
- Uniqueness of the solution needs an additional condition, which is given by

Orthogonality Condition (by Mees)

$$\mathbf{F}^{\mathsf{T}} \delta\boldsymbol{\eta} = 0.$$

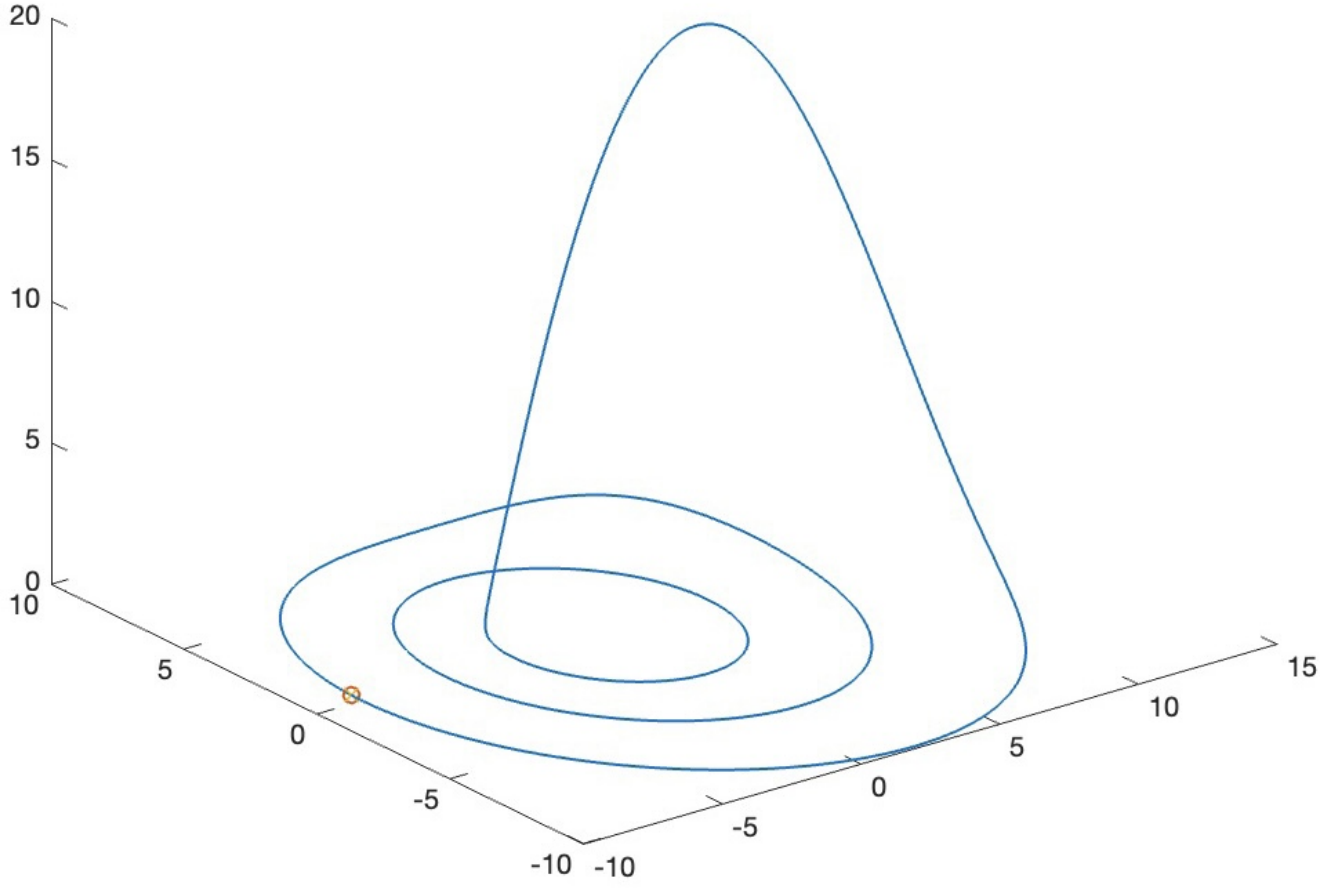
A periodic orbit of the Rössler's System

The defining equations of the Rössler system are:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y - z \tag{7}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + ay \tag{8}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = b + z(x - c) \tag{9}$$


20
15
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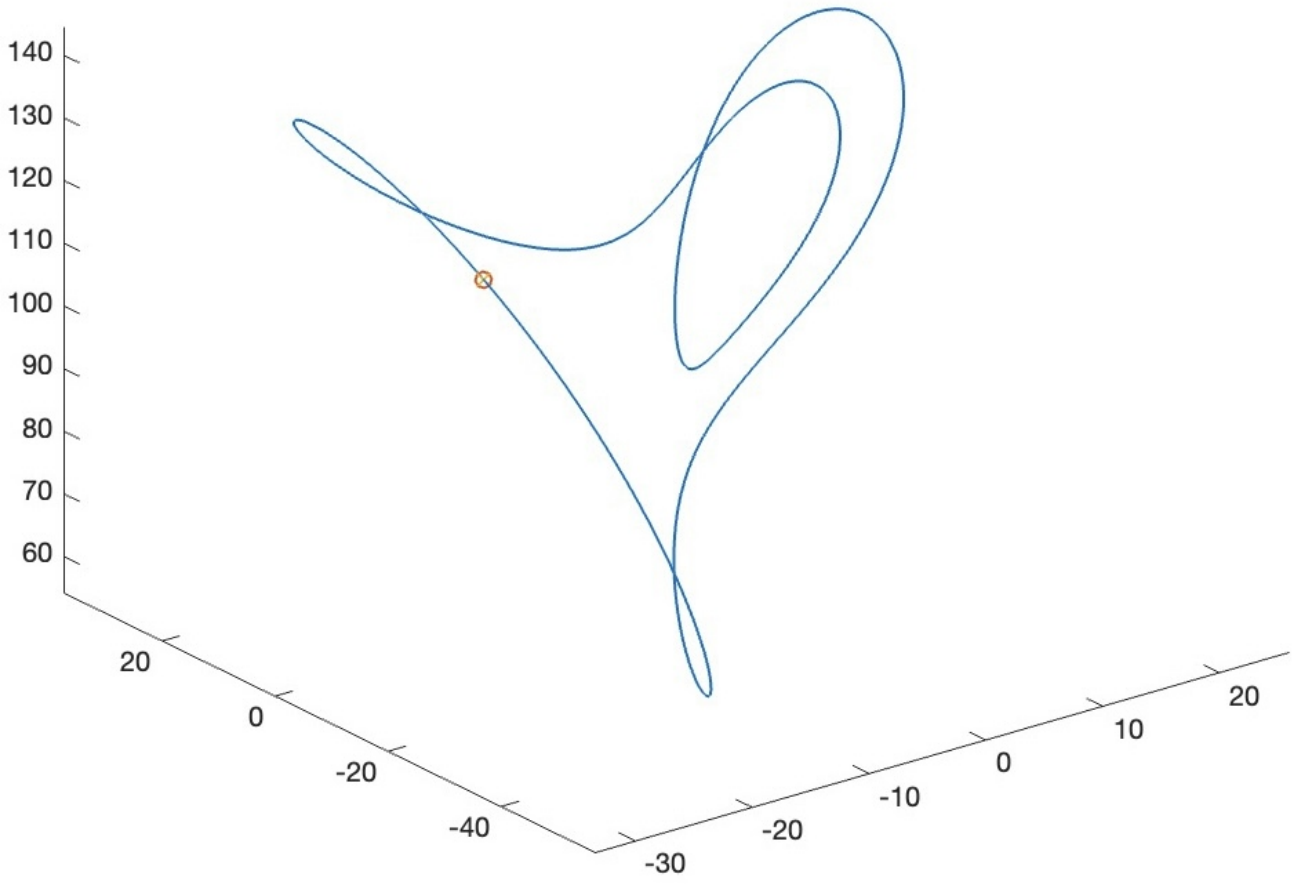
5
10
15

A periodic orbit of the Lorenz System



Continuation of Periodic orbits

- The continuation of periodic orbits is to combine two ideas, a continuation of the equilibria and computing individual periodic orbits.
- The idea is to use a solution $\mathbf{x}[t, \boldsymbol{\eta}(s), \alpha(s)]$ of Equation (2) in order to find the unknown period $T(s)$ such that

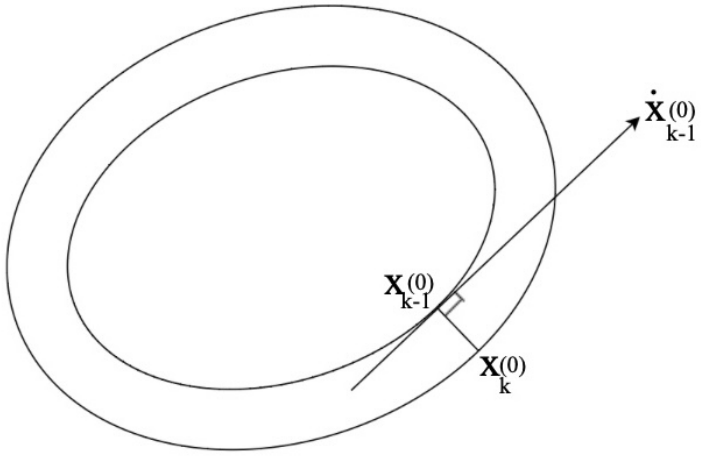
$$\mathbf{x}[T(s), \boldsymbol{\eta}(s), \alpha(s)] = \boldsymbol{\eta}(s).$$

- The system (2) has n equations on $n+2$ unknowns $\boldsymbol{\eta}^{\mathsf{T}}$, T, and α, where T is the transpose.

Conditions for the Uniqueness of the solution

1. The first condition is the pseudo-arclength condition.
2. The second condition can be obtained by making use of the Poincaré orthogonality condition or aligning the phases of the two consecutive periodic solutions.

Poincaré Orthogonality Condition



Poincaré orthogonality condition is defined as enforcing that the change in initial conditions between existing periodic orbit $\mathbf{x}_{k-1}(t)$ and the new periodic orbit $\mathbf{x}_k(t)$ orthogonal to the vector field evaluated at the initial point of $\mathbf{x}_{k-1}(t)$, that is

Poincaré orthogonality condition

$$[\mathbf{x}_k(0) - \mathbf{x}_{k-1}(0)]^{\mathsf{T}} \dot{\mathbf{x}}_{k-1}(0) = 0.$$

Phase Condition

- The most common choice is the integral phase condition, which is obtained by the solution that minimizes

$$D(\omega) := \int_0^T \|\tilde{\mathbf{x}}_k(t+\omega) - \mathbf{x}_{k-1}(t)\|^2 \mathrm{d}t, \qquad (10)$$

  where $\tilde{\mathbf{x}}_k(t+\omega)$ is the solution for any ω.
- Differentiating both sides of Equation (10) with respect to ω, we obtain the minima at $\omega = \omega^*$ (say) , that is,

$$\int_0^T [\tilde{\mathbf{x}}_k(t+\omega^*) - \mathbf{x}_{k-1}(t)]^{\mathsf{T}} \dot{\tilde{\mathbf{x}}}_k(t+\omega^*) \mathrm{d}t = 0.$$

- Let $\tilde{\mathbf{x}}_k(t+\omega^*) \equiv \mathbf{x}_k(t)$, and integrating by parts gives the phase condition

Phase Condition

$$\int_0^T \mathbf{x}_k(t)^{\mathsf{T}} \dot{\mathbf{x}}_{k-1}(t) \mathrm{d}t = \int_0^T \mathbf{x}_k(t)^{\mathsf{T}} \mathbf{F}(\mathbf{x}_{k-1}(t)) \mathrm{d}t = 0.$$

Pseudo-arclength constraint (Condition II)

The second condition is given by the pseudo-arclength constraint (continuation equation)

Pseudo-arclength constraint

$$\int_0^T [\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)]^{\mathsf{T}} \dot{\mathbf{x}}_{k-1}(t) \mathrm{d}t + M = 0,$$

where $M = (T_k - T_{k-1})\dot{T}_{k-1} + (\alpha_k - \alpha_{k-1})\dot{\alpha}_{k-1} - \Delta s.$

Modification for Hamiltonian System

Consider the system

$$\dot{u} = X_{\mathcal{H}}(u) = J\nabla\mathcal{H}, u = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n} \tag{11}$$

and we denote the flow of system by
$\phi_{\mathcal{H}}(t, u) = \phi^t_{\mathcal{H}}(u)$, where J is $2n \times 2n$ matrix.

- In Hamiltonian systems, conserved quantities and symmetries are related by Noether's theorem.
- The absence of the internal parameter in the Hamiltonian systems causes the general continuation scheme to fail.
- Therefore, conserved quantities are used as a continuation parameter to continue the the periodic orbits.

Modified System

- The new modified system is

$$\dot{u} = X_{\mathcal{H}}(u) + \sum_{i=1}^{k} \alpha_i \nabla F_i(u) \tag{12}$$

where $F_i \in \mathcal{F}$ are chosen in such a way that $\{\nabla F_i(p_0), 1 \leq i \leq k\}$ forms a basis for $W := \{\nabla F(p_0) : F \in \mathcal{F}\}$.

- The idea of the continuation of a periodic solution of the system (12) is to look for the solutions of the equation

$$G(T, p, \alpha) := \phi_{\mathcal{H}}(T, p, \alpha) - p = 0.$$

- Let $u(t)$ is a T-periodic solution of the system (12). Then, we have,

$$\frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = \|\nabla F(u(t))\|^2, \tag{13}$$

where $F(u) := \sum_{i=1}^{k} \alpha_i F_i(u)$.

Modified System

- Integrating both of Equation (13) with respect to t, we get

$$\int_0^T \|\nabla F(u(t))\|^2 \mathrm{d}t = F(u(T)) - F(u(0)) = 0 \qquad (14)$$

$$\implies \nabla F(u(t)) = 0, \forall t \in \mathbb{R}.$$

- From Equation (14), $\nabla F_j(p_0)$ are linearly independent $\forall j$.
- If $u(0)$ is taken sufficiently close to p_0, then the same argument is shows that $\nabla F_j(u(0))$ are linearly independent $\forall j$.
- The linearly independency of $\nabla F_j(p_0)$ shows that the system (12) can only have a periodic orbit near Γ_0 (periodic orbit generated by p_0) if $\alpha = 0$.
- If $\alpha = 0$, then the periodic orbits of the system (12) are same as the periodic orbits of the Hamiltonian system (11).

Example: I

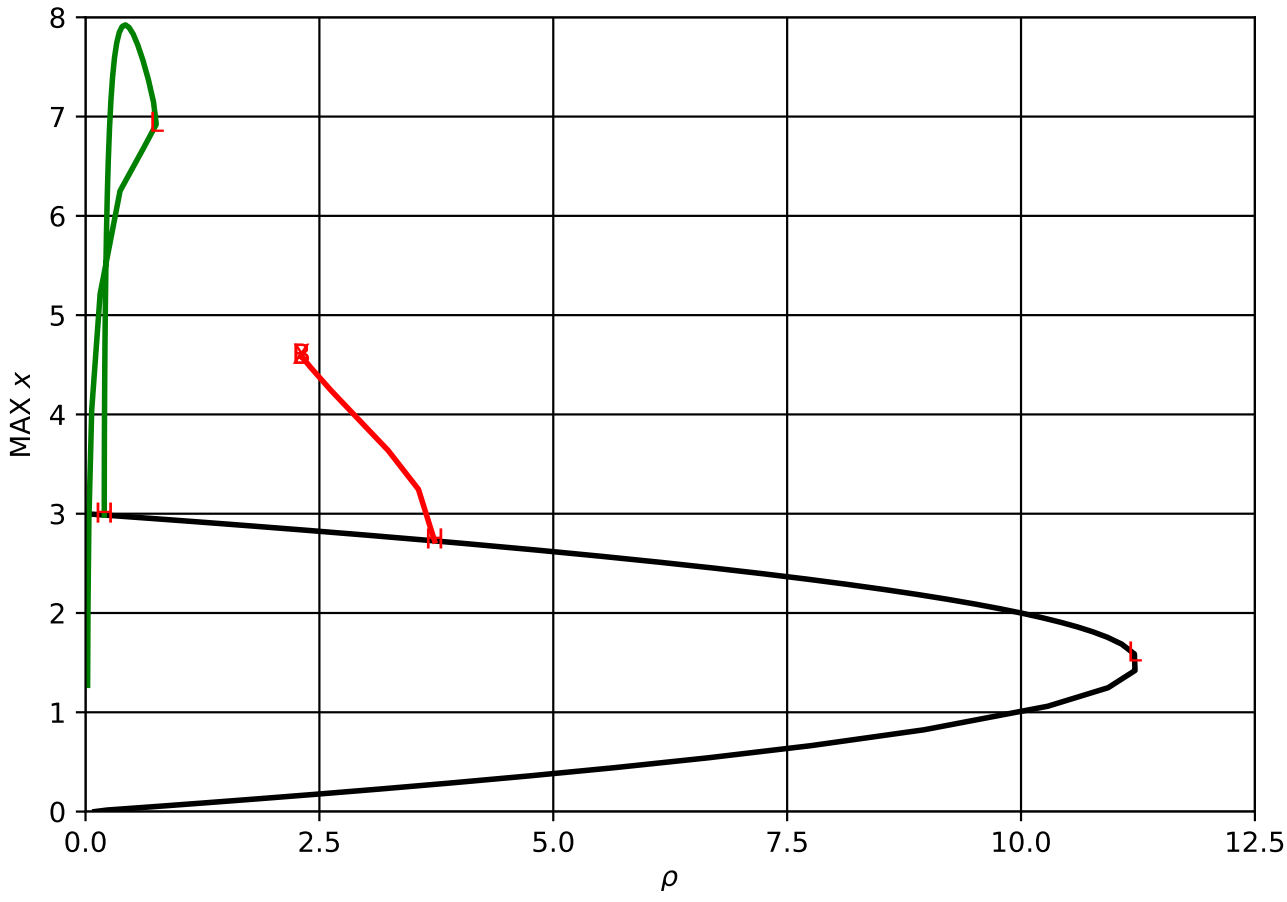


Figure: Bifurcation diagram for Rössler's System

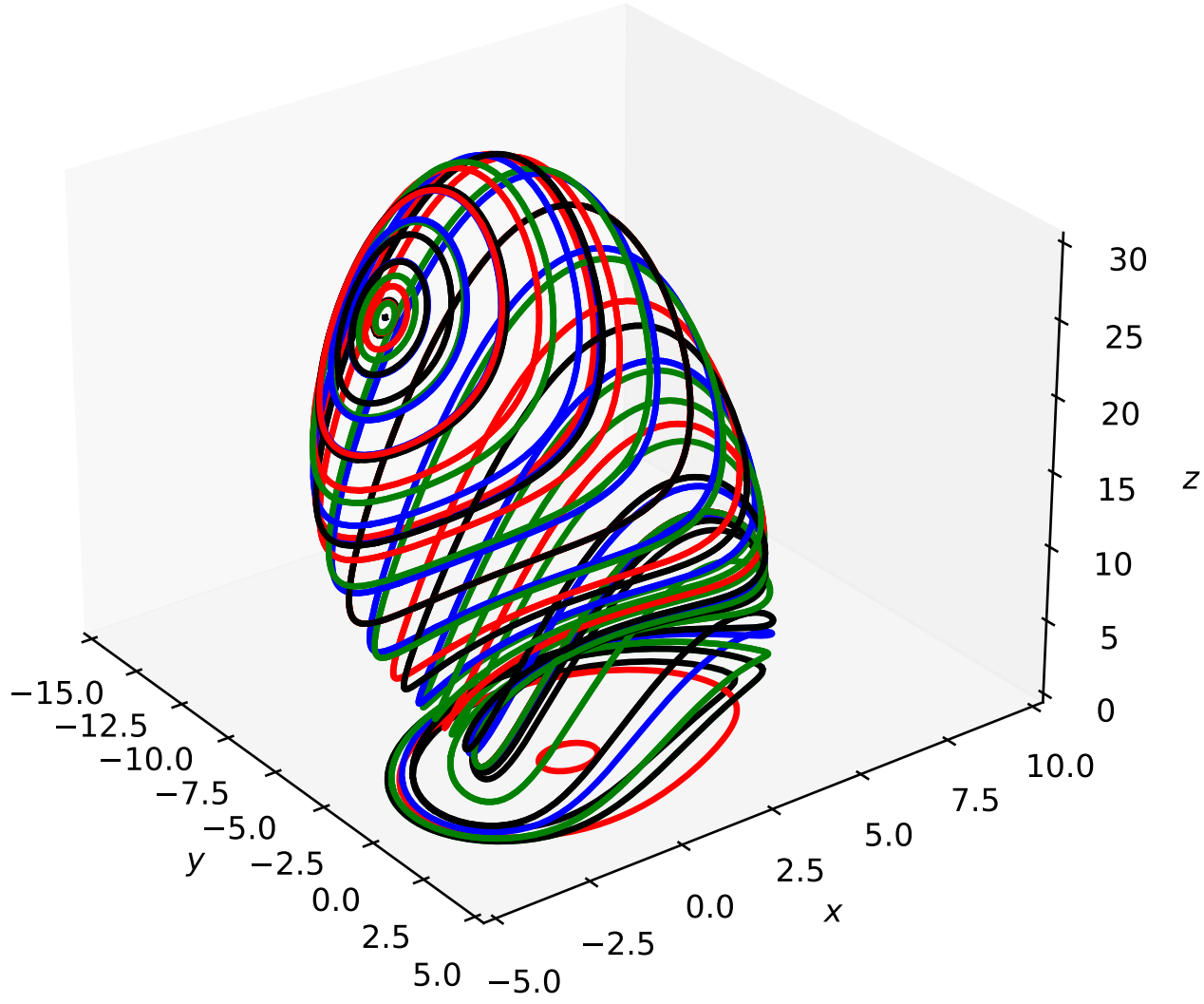


Figure: Continuation of a single periodic orbit in the Rössler's System using AUTO

Example: II

The defining equations of the modified pendulum system are:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 + \lambda \sin(x_1) \tag{15}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\sin(x_1) + \lambda x_2 \tag{16}$$

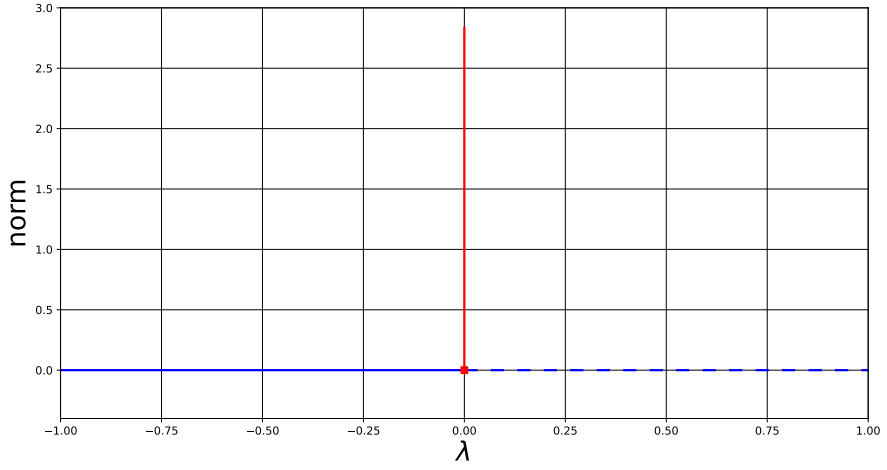


Figure: Bifurcation Diagram for the Modified Pendulum System

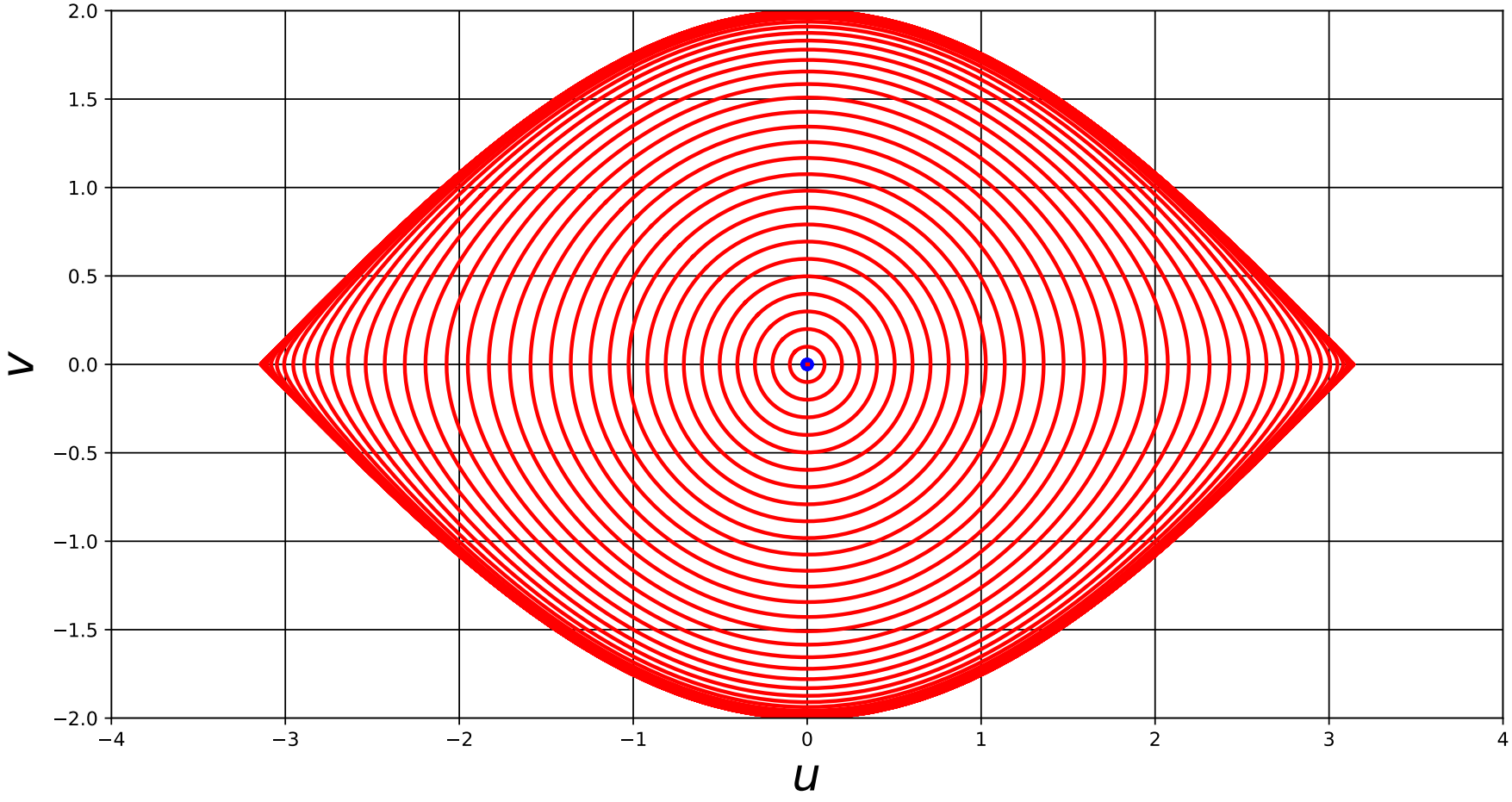


Figure: x vs y solution for the Modified Pendulum System

Example III

Consider the system

$$\begin{cases} m\ddot{x} + \dot{U}(x) + \epsilon \dot{F}(x)a = 0 \\ \ddot{a} + \omega^2 a + \epsilon F(x)a = 0 \end{cases}. \tag{17}$$

with the Hamiltonian

$$\mathcal{H} := \frac{m}{2}\dot{x}^2 + U(x) + \frac{1}{2}(\dot{a}^2 + \omega^2 a^2) + \epsilon F(x)a. \tag{18}$$

We will discuss a special case for which

$$U(x) = e^{-2x} - e^{-x}, F(x) = e^{-x}, m = 1, \epsilon = 1, \omega^2 = 1.5.$$

Writing the above system as the first order ODEs

$$\begin{cases} \dot{x} = y \\ \dot{y} = 2e^{-2x} - e^{-x} + ae^{-x} \\ \dot{a} = z \\ \dot{z} = -1.5a - e^{-x} \end{cases}. \tag{19}$$

with the Hamiltonian

$$\mathcal{H} := \frac{1}{2}\dot{x}^2 + e^{-2x} - e^{-x} + \frac{1}{2}(\dot{a}^2 + 1.5a^2) + e^{-x}a. \tag{20}$$

We have the given system

$$\begin{cases} \dot{x} = y \\ \dot{y} = 2e^{-2x} - e^{-x} + ae^{-x} \\ \dot{a} = z \\ \dot{z} = -2a - e^{-x} \end{cases} \tag{21}$$

with the periodic boundary conditions:

$$x(T) = x(0), a(T) = a(0), \tag{22}$$

for some unknown period T. The system (19) has periodic orbits that are even in time. Therefore, we may look for periodic solutions that satisfy

$$\begin{cases} \dot{x}(0) = y(0) = 0 \\ \dot{a}(0) = z(0) = 0. \end{cases} \tag{23}$$

Single Period using Matlab

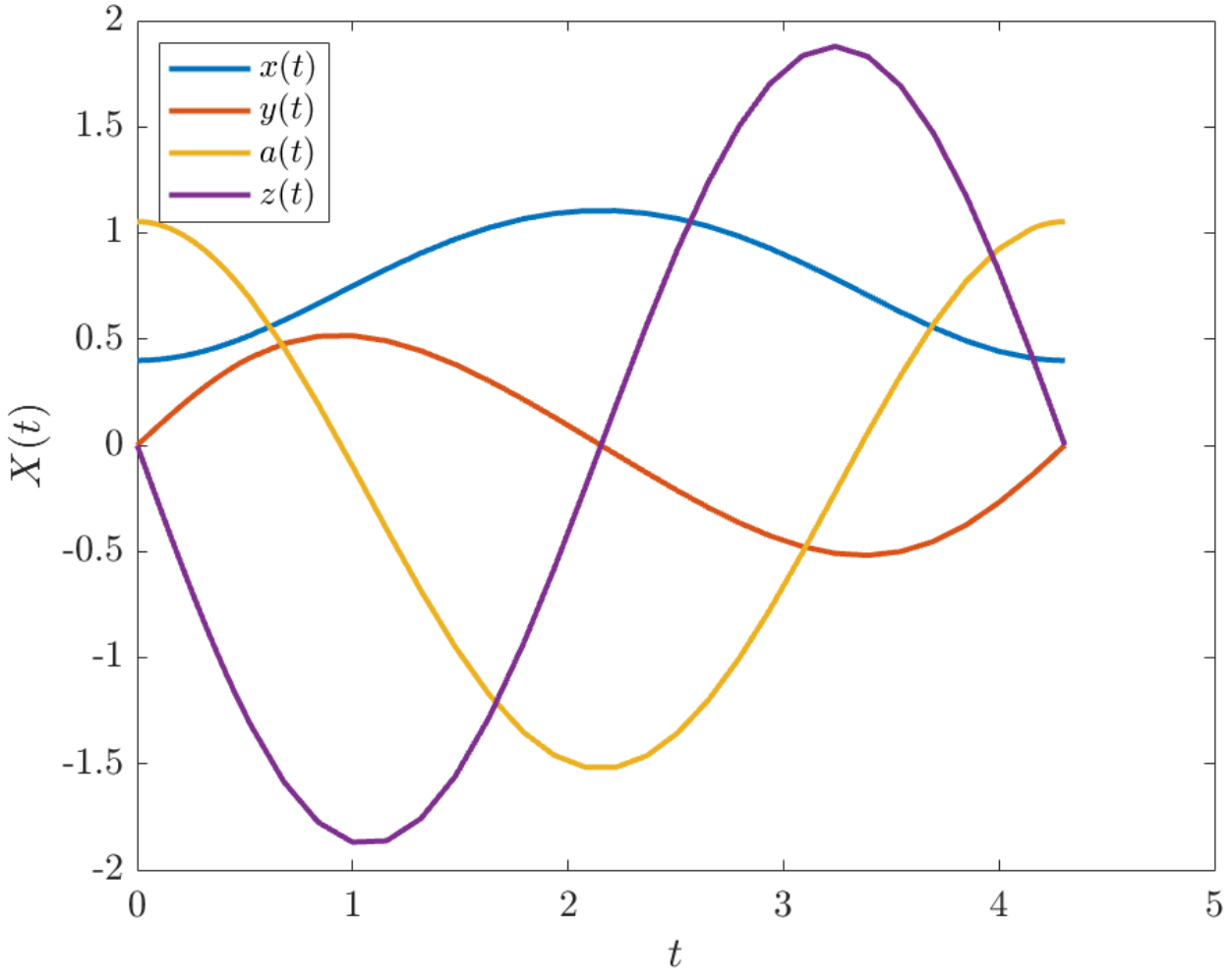


Figure: A single periodic solution for the system 24

- The Hamiltonian modification of the system 19 is given as

$$\dot{x} = \mathcal{J}\nabla\mathcal{H} + \alpha\nabla\mathcal{H}, \alpha \text{ is an unfolding parameter.} \qquad (24)$$

where

$$\mathcal{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

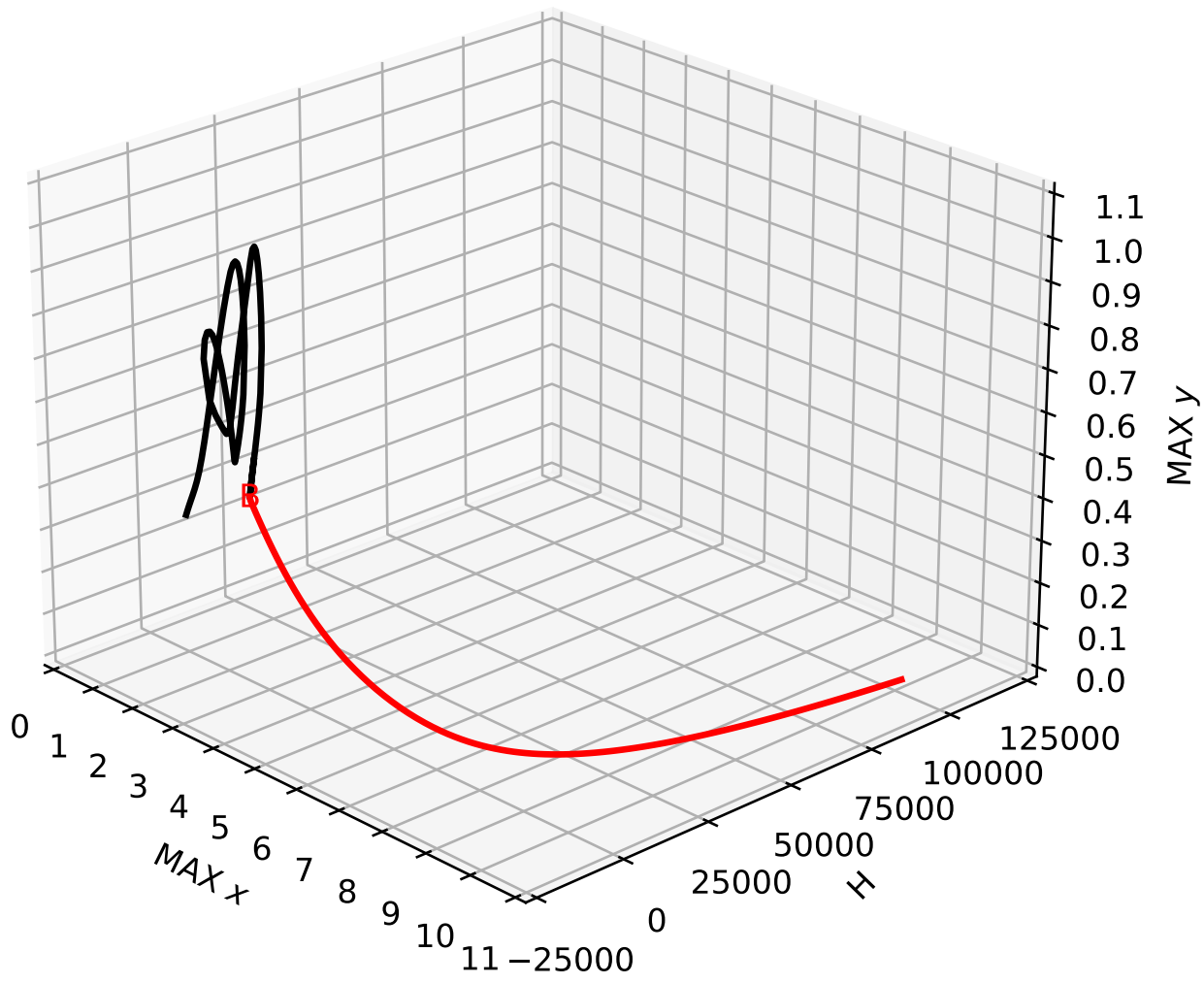


Figure: Plot of conserved quantity with the solutions 24

Future Goals

- The immediate goal is set up the model for the N-vortex system and apply the numerics that we proposed.
- The next goal is to construct and analyze the relative periodic orbits, and understand the dynamics of such vortex model, such as the linear stability of the system.
- We want to study the nonlinear dynamics when the generalized leapfrogging orbit is unstable. For example: Is there a generalization of the walkabout orbit for $N > 2$.

Thank You