Point Vortex Dipole Scattering

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Abstract

We investigate three problems in point-vortices dynamics within a two-dimensional, inviscid, incompressible fluid. We derive a new reduction of a system of three vortices. The integrable reduced system has an easily visualized phase plane that illuminates the dynamics. We apply it to explain the scattering of the point-vortex dipole with a third vortex in two cases. We then add a fourth vortex and use the reduced dynamics of the three-vortex system as the basis for the perturbative study of dipole-dipole scattering.

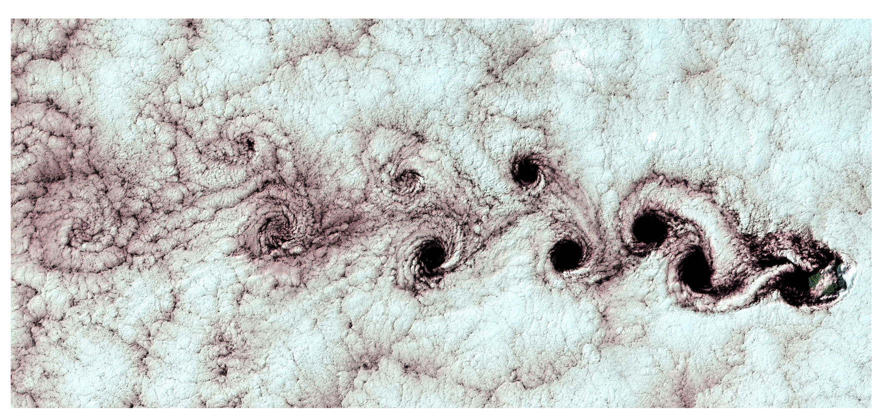
The Point Vortex Model

The N vortex positions satisfy [4]:

$$\dot{x}_i = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{|\mathbf{r}_i-\mathbf{r}_j|^2}(y_i-y_j), \qquad \dot{y}_i = \frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{|\mathbf{r}_i-\mathbf{r}_j|^2}(x_i-x_j).$$

with the conserved Hamiltonian [5, 7],

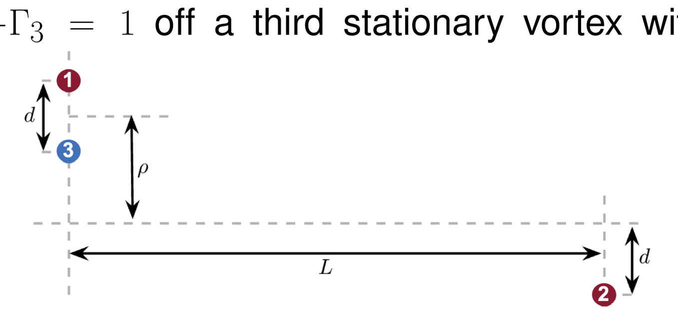
$$\mathcal{H}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\cdots,\mathbf{r}_N) = -\frac{1}{4\pi}\sum_{1\leq i<j\leq N}\Gamma_i\Gamma_j\log\|\mathbf{r}_i-\mathbf{r}_j\|^2.$$



Vortices in the atmosphere.

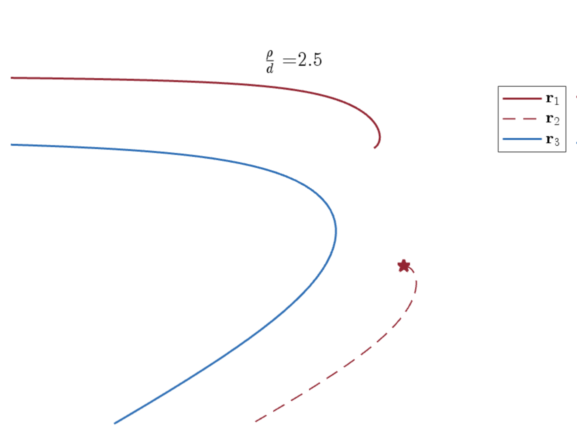
The Scattering Problem

Aref [1] considered the scattering of a vortex dipoles with circulations $\Gamma_1 = -\Gamma_3 = 1$ off a third stationary vortex with $\Gamma_2 = 1$.

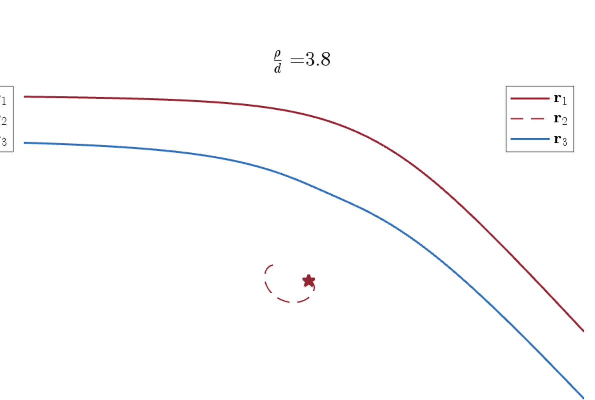


The system exhibits the following Scattering behavior:

- **Direct Scattering** (Large $|\rho|$): The original dipole interacts briefly with vortex 2 before escaping infinity.
- **Exchange Scattering** (Small $|\rho|$): A new dipole consisting of vortices 2 and 3 escapes to infinity.

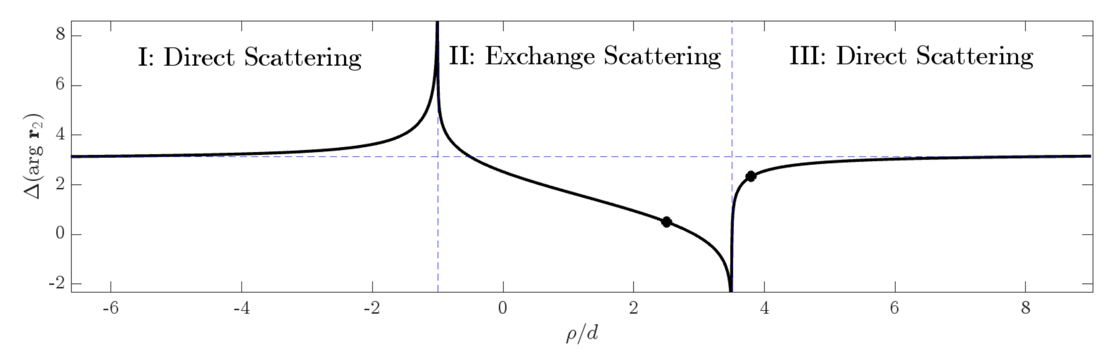


Exchange Scattering



Direct Scattering

Following Lydon et al.[6],



Previous studies dating back to Gröbli [2, 3] use a coordinate system based on the side lengths of a triangle with its vertices located at the three vortices.

This coordinate system imposes limitations

- The equations are algebraically unwieldy and are singular at all collinear configurations.
- The reduction introduces nonphysical singularities.
- Understanding scattering requires solving the ODE system with elliptic functions instead of phase plane insights.

Jacobi Coordinates and Nambu Dynamics

The Jacobi coordinate transformation is used to simplify the formulation in n-body problems by replacing the coordinates of two vortices at positions $\mathbf{r}_j$ and $\mathbf{r}_{j+1}$ by their displacement $\mathbf{R}_j = \mathbf{r}_{j+1} - \mathbf{r}_j$ and their center of vorticity $\tilde{\mathbf{r}}_{j+1} = \frac{\Gamma_j\mathbf{r}_j+\Gamma_{j+1}\mathbf{r}_{j+1}}{\Gamma_j+\Gamma_{j+1}}$, and then repeating the process. The final Hamiltonian is independent of the center of vorticity $\mathbf{R}_3$. After one more normalization

$$H = -\frac{1}{2}\log\|\mathbf{R}_1\|^2 + \frac{1}{2}\log\left\|\mathbf{R}_2+\frac{\mathbf{R}_1}{2}\right\|^2 + \frac{1}{2}\log\left\|\mathbf{R}_2-\frac{\mathbf{R}_1}{2}\right\|^2,$$

We further simplify the dynamics using Nambu brackets, which reformulate mechanics naturally when there are two conserved quantities. According to the Nambu dynamics,

$$\dot{F} = \{F, 2\Theta^2, H\}, \text{ where } \{F,G,K\} = \nabla F\cdot(\nabla G\times\nabla K).$$

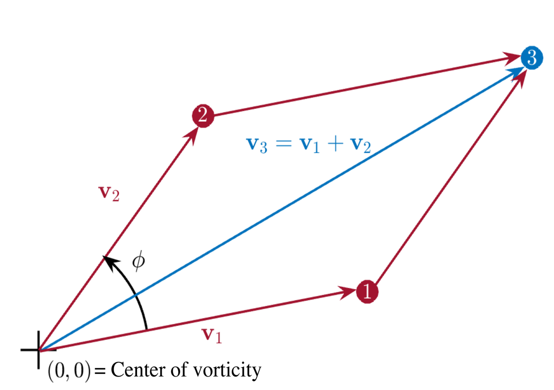
and for the conserved quantities

$$H = -\frac{1}{2}\log(Z+\Theta) + \frac{1}{2}\log\left(Z^2-X^2\right) \text{ and } \Theta^2 = Z^2 - X^2 - Y^2,$$

the evolution equations are:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{-2Y}{Z+\Theta} + \frac{4ZY}{Z^2-X^2};$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{2X}{Z+\Theta};$$
$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{4XY}{Z^2-X^2},$$

with the new coordinates:

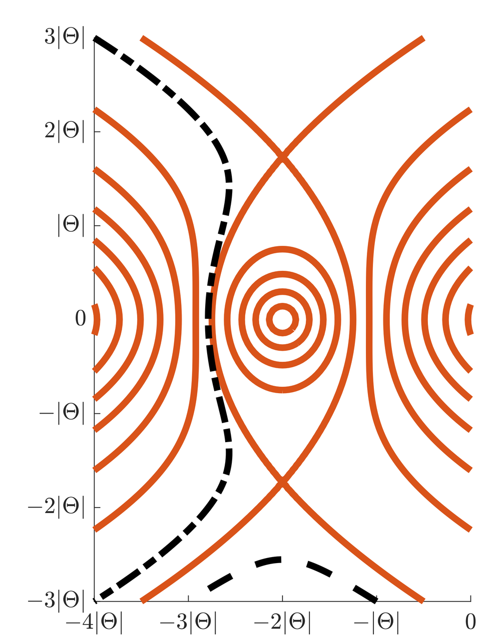


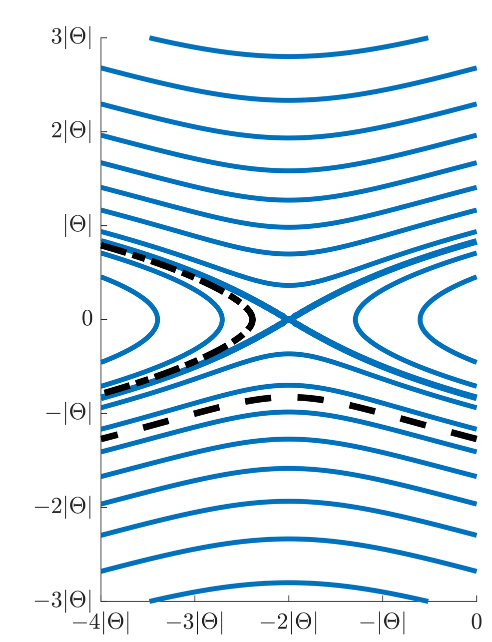
$$X = -\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2;$$
$$Y = 2\|\mathbf{v}_1\|\|\mathbf{v}_2\|\sin\phi;$$
$$Z = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2;$$
$$\Theta = -2\|\mathbf{v}_1\|\|\mathbf{v}_2\|\cos\phi.$$

Scattering for the case $\Gamma_2 = 1$

In the $X-Y$ coordinate system, initial conditions leading to each type of scattering can be identified simply using phase-plane reasoning.

As ρ is varied Θ so does Θ and the trajectory on the phase plane changes at values where the scattering diagram has vertical asymptotes, trajectories on a stable manifold of a hyperbolic fixed point.



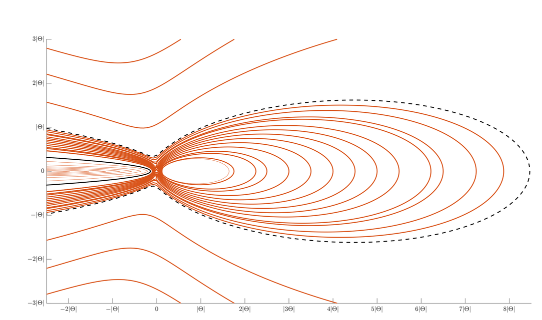


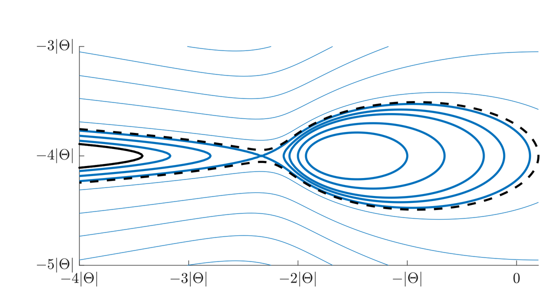
Interpreting the dynamics using the phase planes

- As $t\to-\infty$ trajectories retreat to infinity in the fourth quadrant.
- Exchange scattering solutions map to the dashed trajectories.
- Direct scattering solutions map to the dash-dot trajectories.
- Singular values of ρ in the scattering diagram correspond to the stable manifolds of hyperbolic points in the phase plane. This explains the critical values $\frac{\rho}{d} = -1$ and $\frac{\rho}{d} = \frac{7}{2}$.

Scattering for the case $\Gamma_2 \neq 1$

A Question: When $\Gamma_2 \neq 1$, exchange scattering is no longer possible since vortices 2 and 3 cannot form a dipole. What replaces it?





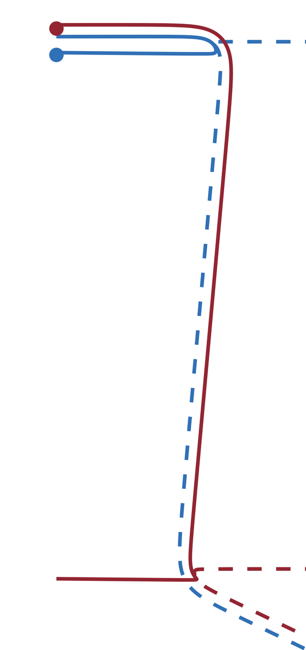
An Answer:

- For $\Gamma \neq 1$ a new elliptic fixed point or singular point appears on the right of the origin, surrounded by a homoclinic loop.
- All trajectories that begin in the fourth quadrant below the separatrices as $t\to-\infty$ now wrap around the closed homoclinic and escape to infinity in the third quadrant. Thus, exchange scattering solutions get replaced by long-path direct scattering solutions.

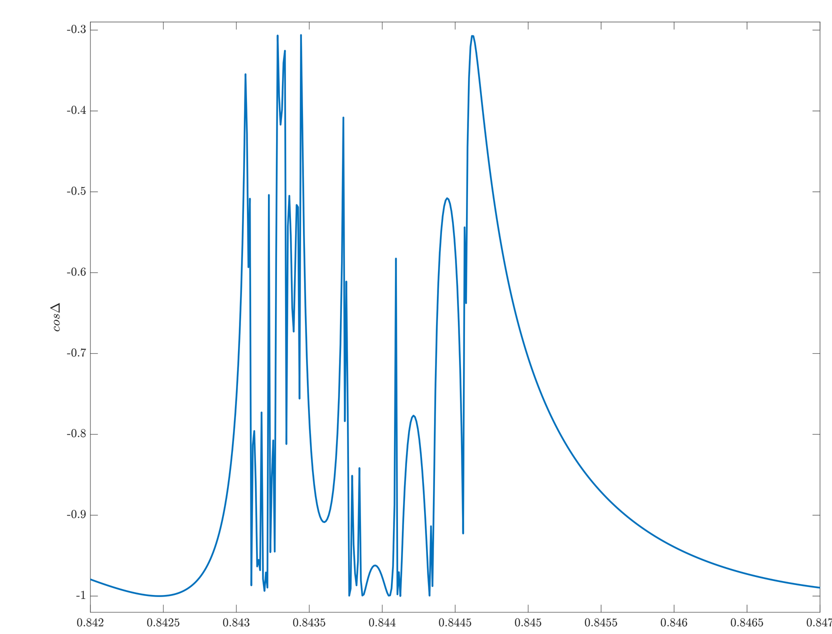
Dipole-Dipole Scattering

Now consider the scattering of a narrow, fast, dipole that collides with a wide, slow dipole.

- The simulation on the left shows two vortex-dipole scattering events separated by intervals of free motion.
- Sweeping over parameters demonstrates *chaotic scattering*: the output angle shows sensitive dependence on the input parameter.



Exchange-Exchange scattering



Chaotic Scattering

The Hamiltonian for the four-vortex system is, in Jacobi coordinates,

$$H = -\frac{1}{2}\log\|\mathbf{R}_1\|^2 + \frac{1}{2}\log\left\|\mathbf{R}_2+\frac{\mathbf{R}_1}{2}\right\|^2 + \frac{1}{2}\log\left\|\mathbf{R}_2-\frac{\mathbf{R}_1}{2}\right\|^2$$
$$+\frac{1}{2}\log\left\|\frac{1}{2}\mathbf{R}_1-\mathbf{R}_2+\mathbf{M}\right\|^2 + \frac{1}{2}\log\left\|-\frac{1}{2}\mathbf{R}_1-\mathbf{R}_2+\mathbf{M}\right\|^2$$
$$-\frac{1}{2}\log\|-2\mathbf{R}_2+\mathbf{M}\|^2$$

where $\mathbf{M} = \mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4$ is the conserved linear impulse.

The new terms can be treated perturbatively when the fourth vortex is far from the other three.

Acknowledgements

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