Global Phase Plane Analysis of Three Vortex Interactions

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Abstract

We investigate two problems in point-vortex dynamics in a two-dimensional, inviscid, incompressible fluid. We derive a novel reduction of a system involving three vortices, initially employing Jacobi coordinates followed by Nambu brackets. First, we conduct a global phase analysis of a three-vortex problem with arbitrary circulations. Second, we generalize the reduction method to study the dynamics of four vortices with vanishing total circulation. The novel reduction method eliminates coordinate singularities that made understanding the dynamics challenging.

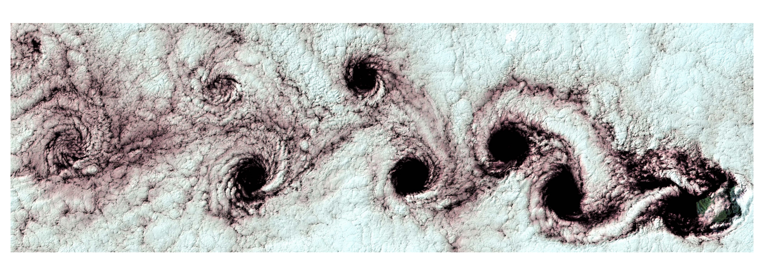
The Point Vortex Model

The N vortex positions satisfy (5):

$$\dot{x}_i = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{|\mathbf{r}_i-\mathbf{r}_j|^2}(y_i-y_j), \qquad \dot{y}_i = \frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{|\mathbf{r}_i-\mathbf{r}_j|^2}(x_i-x_j).$$

with the conserved Hamiltonian (6, 7),

$$\mathcal{H}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\cdots,\mathbf{r}_N) = -\frac{1}{4\pi}\sum_{1\le i<j\le N}\Gamma_i\Gamma_j\log\left\|\mathbf{r}_i-\mathbf{r}_j\right\|^2.$$



Vortices in the atmosphere.

Previous Studies and Limitations

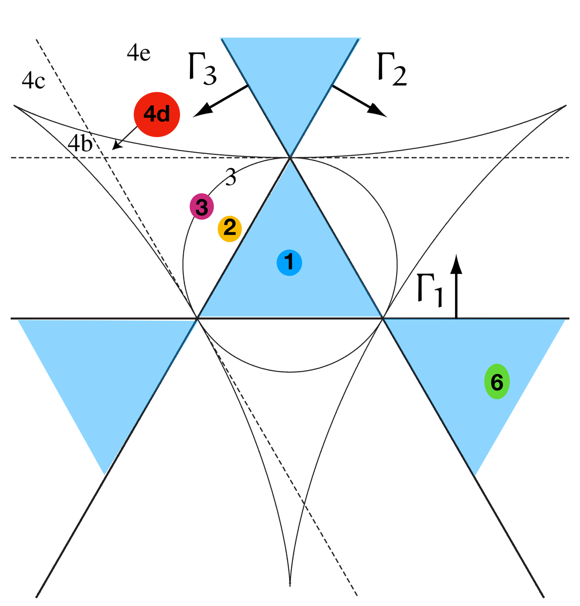
Previous studies by Gröbli (1, 4) introduced a coordinate system based on the triangle side lengths with vertices at the three vortices. The coordinate system has the following issues:

- Singularity in equations at collinear configurations.
- Nonphysical singularities introduced during reduction.

Under the assumption

$$\Gamma_1+\Gamma_2+\Gamma_3 = 1,$$

Aref derived a bifurcation diagram showing how the phase space depends on the circulations.

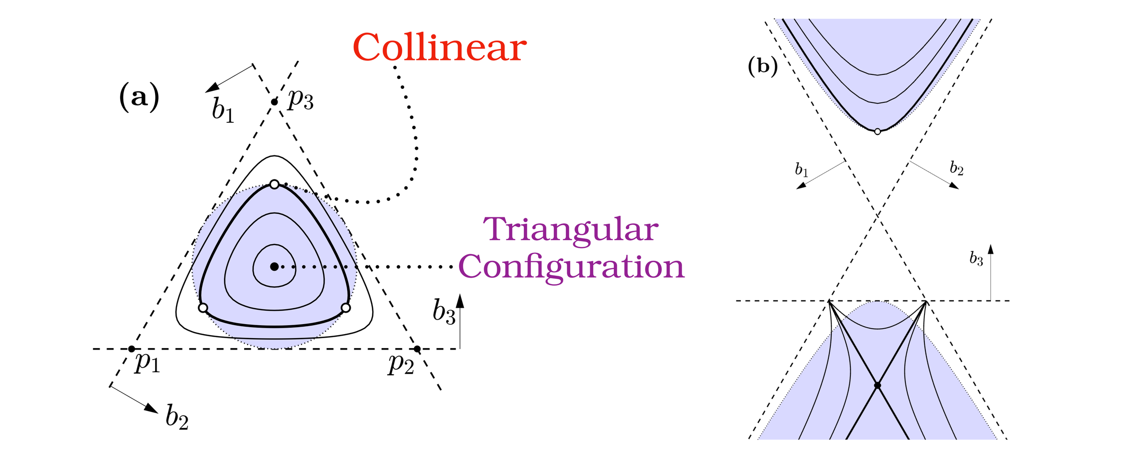


Aref's barycentric coordinates illustrate how three-vortex dynamics vary with the sign of κ_2: blue shading shows spherical dynamics ($\kappa_2>0$), while the unshaded area represents hyperbolic dynamics ($\kappa_2<0$) in the XY plane projection.

Aref's Phase Planes

- Portions lying outside the shaded regions lack physical meaning.
- Dynamics singular at collinear relative equilibria (∘).
- Aref's phase space based on his method is hard to follow.

[1] A famous example of the *vortex collapse* case, where $\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_1\Gamma_3=0$.
[2] The dipole is formed by same vortices after interacting with initial vortex 2; else Exchange.
[3] Region 4 (d) in 4.



Vortices of circulation $(1,1,1)$ Vortices of circulation $(1,1,-1)$

Phase diagrams in trilinear coordinates for vortices of circulations $(1,1,-1)$ in Aref's trilinear coordinates.

A novel coordinate reduction
I. Jacobi Coordinates

- The Jacobi coordinate transformation is used to simplify the formulation in n-body problems.
- It replaces the coordinates of two vortices at positions $\mathbf{r}_j$ and $\mathbf{r}_{j+1}$ by their displacement $\mathbf{R}_j = \mathbf{r}_{j+1}-\mathbf{r}_j$ and their center of vorticity. The process is applied iteratively.

Jacobi coordinates $\mathbf{R}$ for three vortices with corresponding *reduced circulations* κ are defined as:

$$\mathbf{R}_1 = \mathbf{r}_1-\mathbf{r}_2; \qquad \kappa_1 = \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2},$$
$$\mathbf{R}_2 = \frac{\Gamma_1\mathbf{r}_1+\Gamma_2\mathbf{r}_2}{\Gamma_1+\Gamma_2}-\mathbf{r}_3; \qquad \kappa_2 = \frac{(\Gamma_1+\Gamma_2)\Gamma_3}{\Gamma_1+\Gamma_2+\Gamma_3},$$

where $\mathbf{r}_i$ are the vortex positions. WLOG, the center of vorticity can be placed at the origin. We assume

$$\Gamma_1 \ge \Gamma_2 > 0.$$

Sign Conditions:

$$\kappa_1>0; \quad \kappa_2 = \begin{cases} >0, & \text{if } \Gamma_3>0 \text{ or } \Gamma_3<-\Gamma_1-\Gamma_2; \\ <0, & \text{if } -\Gamma_1-\Gamma_2<\Gamma_3<0. \end{cases}$$

II. Nambu Dynamics

We use *Nambu brackets* for reformulating dynamics:

$$\dot{F} = \{F,\Theta^2,H\}, \quad \{F,G,K\} = \nabla F\cdot(\nabla G\times\nabla K),$$

where the conserved quantity Θ is defined by:

$$\Theta^2 = \begin{cases} Z^2-X^2-Y^2, & \text{if } \kappa_2<0, \\ Z^2+X^2+Y^2, & \text{if } \kappa_2>0. \end{cases}$$

Hamiltonian and Angular Impulse in Jacobi-Nambu

In Jacobi coordinates, the Hamiltonian H and angular impulse Θ are:

$$H = -\frac{\Gamma_1\Gamma_2}{2}\log\|\mathbf{R}_1\|^2 - \frac{\Gamma_2\Gamma_3}{2}\log\left\|\mathbf{R}_2-\frac{\kappa_1}{\Gamma_2}\mathbf{R}_1\right\|^2 - \frac{\Gamma_1\Gamma_3}{2}\log\left\|\mathbf{R}_2+\frac{\kappa_1}{\Gamma_1}\mathbf{R}_1\right\|^2.$$

$$\Theta = \kappa_1\|\mathbf{R}_1\|^2+\kappa_2\|\mathbf{R}_2\|^2.$$

Dynamics Based on κ_2 Sign

- $\kappa_2<0$: Represents a **two-sheeted hyperboloid** in (X,Y,Z,Θ) coordinates.
- $\kappa_2>0$: Represents a **sphere** in (X,Y,Z,Θ) coordinates.

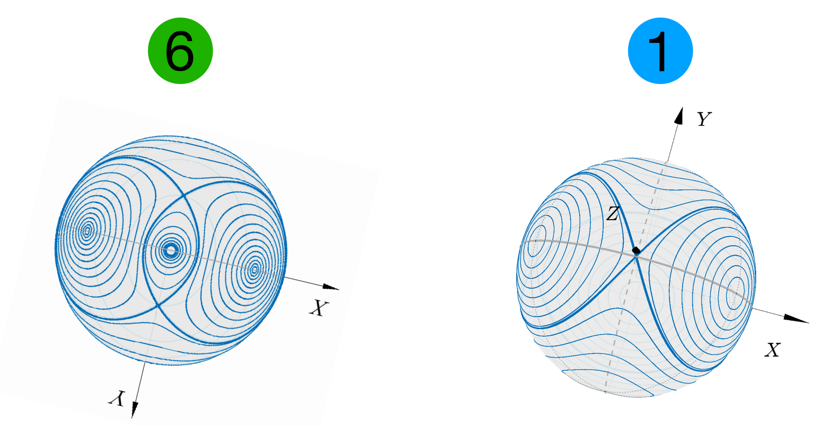
Evolution Equations

The system's dynamics are given by:

$$\frac{dX}{dt} = 4H_ZY, \quad \frac{dY}{dt} = -4H_ZX+4H_XZ, \quad \frac{dZ}{dt} = -4H_XY.$$

Two Equal Vortices: $(\Gamma,\Gamma,1-2\Gamma),\ \Gamma>0$

Two Spherical Cases

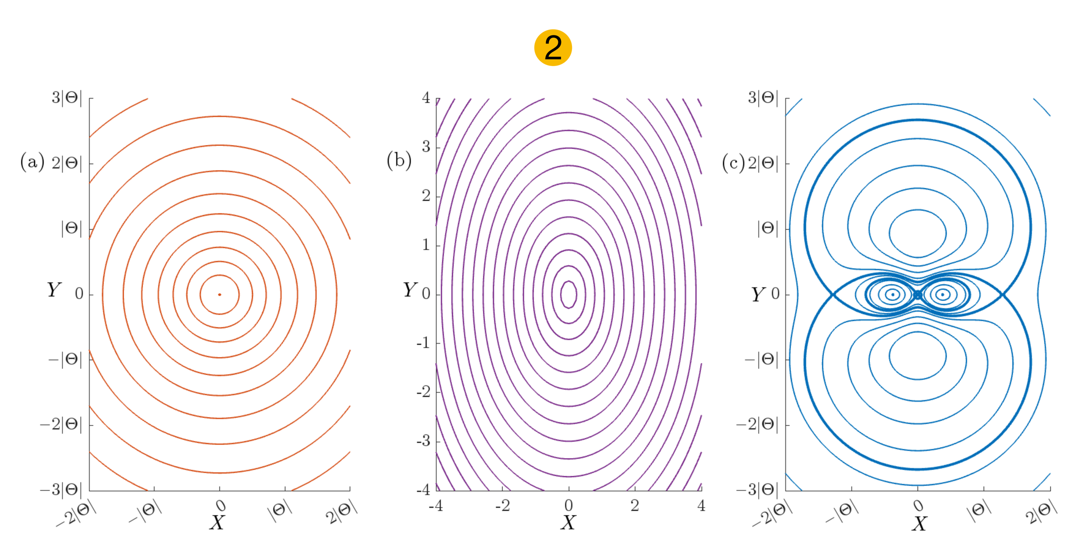


$(\Gamma_1,\Gamma_2,\Gamma_3) = (-2,-2,5)$ $(\Gamma_1,\Gamma_2,\Gamma_3) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

The phase-sphere of the three-vortex system with $\Gamma=-2$ and $\Gamma=\frac{1}{3}$.

- From Region 6 to Region 1: Equilibrium points change from centers to saddles when vortices move from Region 6 to Region 1.
- The equator $Y=0$, representing collinear vortex configurations, and the meridians, corresponding to isosceles triangle formations.
- Our coordinate system's phase planes are easier to understand and visualize when the vortices are collinear.

An example from Region 2: $\Gamma=\frac{5}{9}$

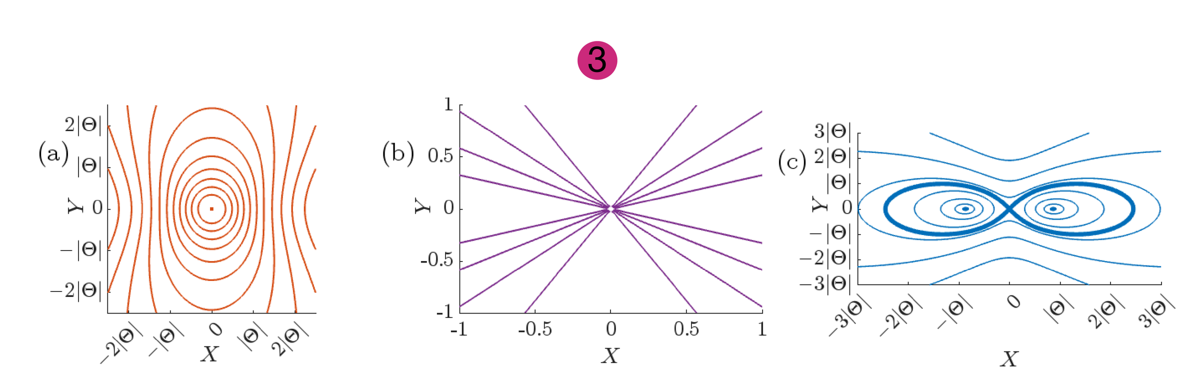


Phase planes for $\Gamma=\frac{5}{9}$.

From Region 2 to Region 3:

- $\Theta<0$, the nature of the singularity does not change.
- For $\Theta=0$, the families of periodic orbits for $\Gamma=\frac{5}{9}$ collapses when $\Gamma=\frac{2}{3}$.
- For $\Theta>0$, the equilibria intersecting at the separatrix for $\Gamma=\frac{5}{9}$ goes off to infinity when $\Gamma=\frac{2}{3}$.

Vortex Collapse[1]: $\Gamma=\frac{2}{3}$

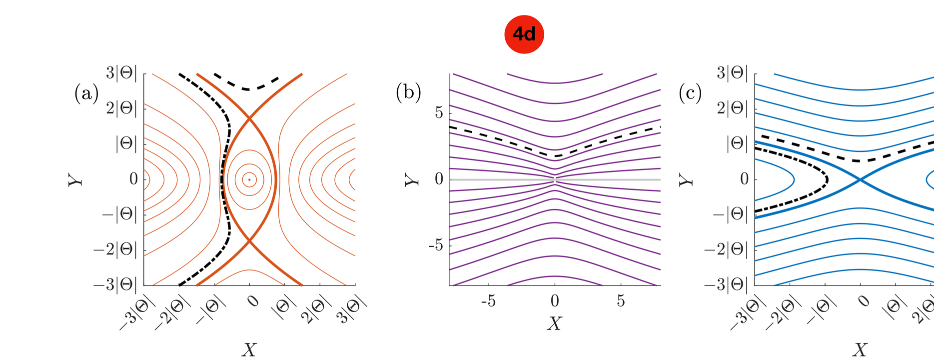


The phase planes for vortex-collapse at $\Gamma=\frac{2}{3}$.

From Region 3 to Region 4 (d):

- $\Theta<0$, the nature of the singularity at the origin does not change, but new equilibria appears at the separatrix.
- For $\Theta=0$, the line $Y=0$ is singular from $\Gamma=\frac{2}{3}$ to $\Gamma=1$.
- For $\Theta>0$, the equilibria at the separatrix go off to $\pm\infty$, leaving with one equilibrium (collinear state) at the origin.
- The phase planes shows the Direct scattering[2] (dash-dot) and Exchange Scattering (dash).

Vortex-Dipole Scattering Problem[3]: $\Gamma=1$



The XY phase planes of system when $(\Gamma_1,\Gamma_2,\Gamma_3)=(1,1,-1)$. (a) The case $\Theta<0$ with singularity (point) and triangular configurations at the intersections of the thick curves. (b) The case $\Theta=0$. The gray line $Y=0$ is singular. (c) the case $\Theta>0$ with collinear equilibrium at the separatrix intersection.

Four-vortex interaction with zero net circulation

Coordinate systems (2, 3) reduce the four-vortex problem to a three-vortex problem. Integrability conditions for four interacting point vortices are:

- The linear impulse of the system is zero under the assumption that $\Gamma_1+\Gamma_2+\Gamma_3\neq 0$.
- The net circulation of the four vortices is zero, i.e, $\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4=0$.

We use Jacobi coordinates for the four-vortex system, normalize it, and obtain the Hamiltonian for the system.

$$\begin{aligned} H = &-\frac{\Gamma_1\Gamma_2}{2}\log\left(\frac{\Theta+Z}{2\kappa_1}\right) - \frac{\Gamma_1\Gamma_3}{2}\log\left(-\frac{Z-\Theta}{2\kappa_2}+\frac{X}{\Gamma_1}\sqrt{-\frac{\kappa_1}{\kappa_2}}+\frac{\kappa_1(\Theta+Z)}{2\Gamma_1^2}\right) \\ &-\frac{\Gamma_2\Gamma_3}{2}\log\left(-\frac{Z-\Theta}{2\kappa_2}-\frac{X}{\Gamma_2}\sqrt{-\frac{\kappa_1}{\kappa_2}}+\frac{\kappa_1(\Theta+Z)}{2\Gamma_2^2}\right) \\ &-\frac{\Gamma_1\Gamma_4}{2}\log\left(-\frac{Z-\Theta}{2\Gamma_2^2\kappa_2}+\frac{X}{\Gamma_2}\sqrt{-\frac{\kappa_2}{\kappa_1}}+\frac{\Theta+Z}{2\kappa_1}\right) \\ &-\frac{\Gamma_2\Gamma_4}{2}\log\left(-\frac{Z-\Theta}{2\Gamma_1^2\kappa_2}-\frac{X}{\Gamma_1}\sqrt{-\frac{\kappa_2}{\kappa_1}}+\frac{\Theta+Z}{2\kappa_1}\right) - \frac{\Gamma_3\Gamma_4}{2}\log\left(-\frac{Z-\Theta}{2\Gamma_3^2\kappa_2}\right). \end{aligned}$$

Acknowledgements

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