The Phase Space of the Three-Vortex Problem and its Application to Vortex-Dipole Scattering

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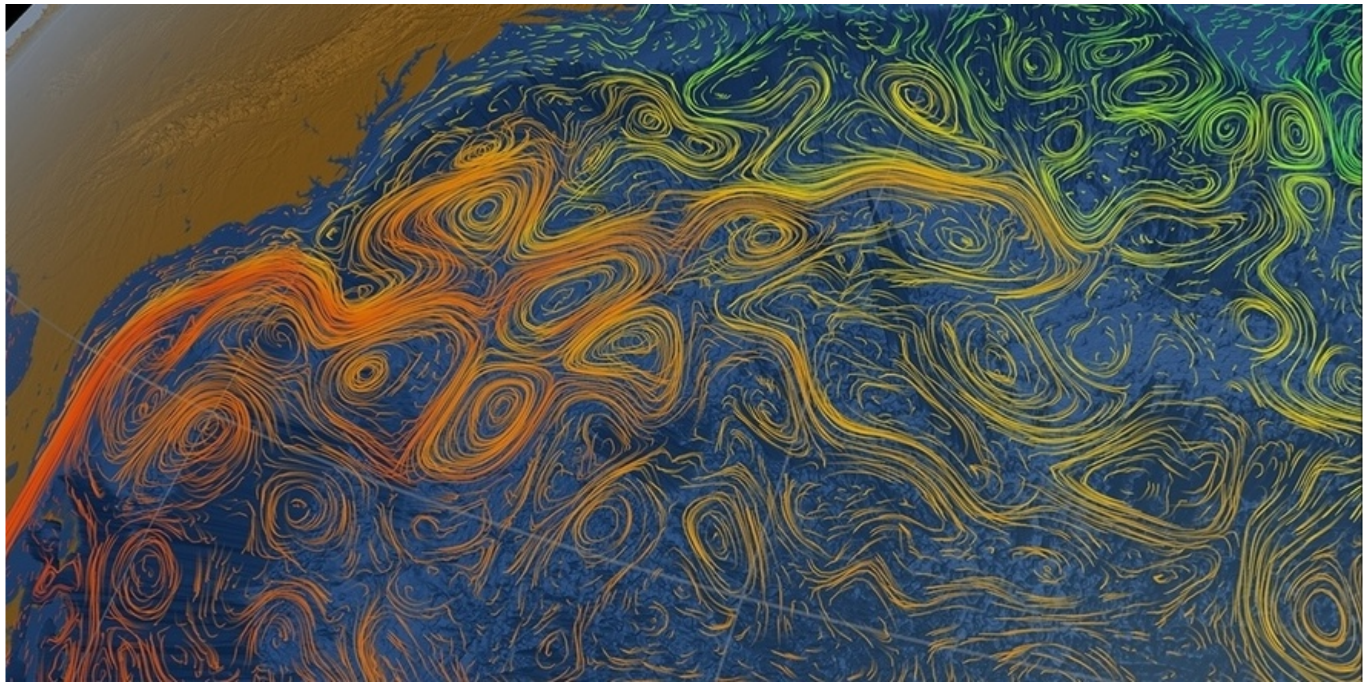
Point Vortices

- A point vortex is a solution to the 2D Euler equations with vorticity concentrated at a single point.

- It is characterized by two quantities: circulation (strength and orientation) and position.

- This model, derived by Helmholtz, idealizes Euler's equations for the motion of interacting point vortices.

- It is a Hamiltonian system, enabling the use of Hamiltonian theory and tools for its study.



Vortices formation while oceans and atmospheres move heat around on Earth and other planetary bodies.

Kirchhoff's Hamiltonian Representation

Consider N-vortices located at $r_i = (x_i, y_i)$ with the strength Γ_i, then the system of ODEs describing the N-vortex motion can be described by the Hamiltonian,

$$\mathcal{H}(r_1, r_2, r_3, \cdots, r_N) = -\frac{1}{4\pi} \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j \log \|r_i - r_j\|^2.$$

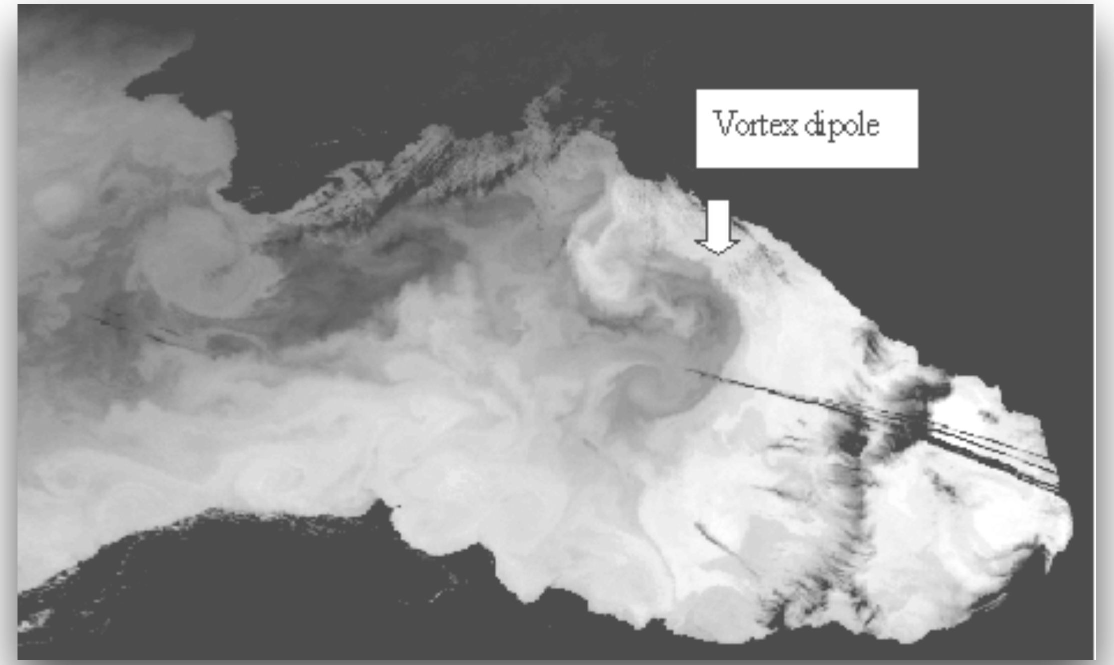
System of 2N point vortex equations:

$$\Gamma_i \frac{dx_i}{dt} = \frac{\partial \mathcal{H}}{\partial y_i}, \qquad \Gamma_i \frac{dy_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i}.$$

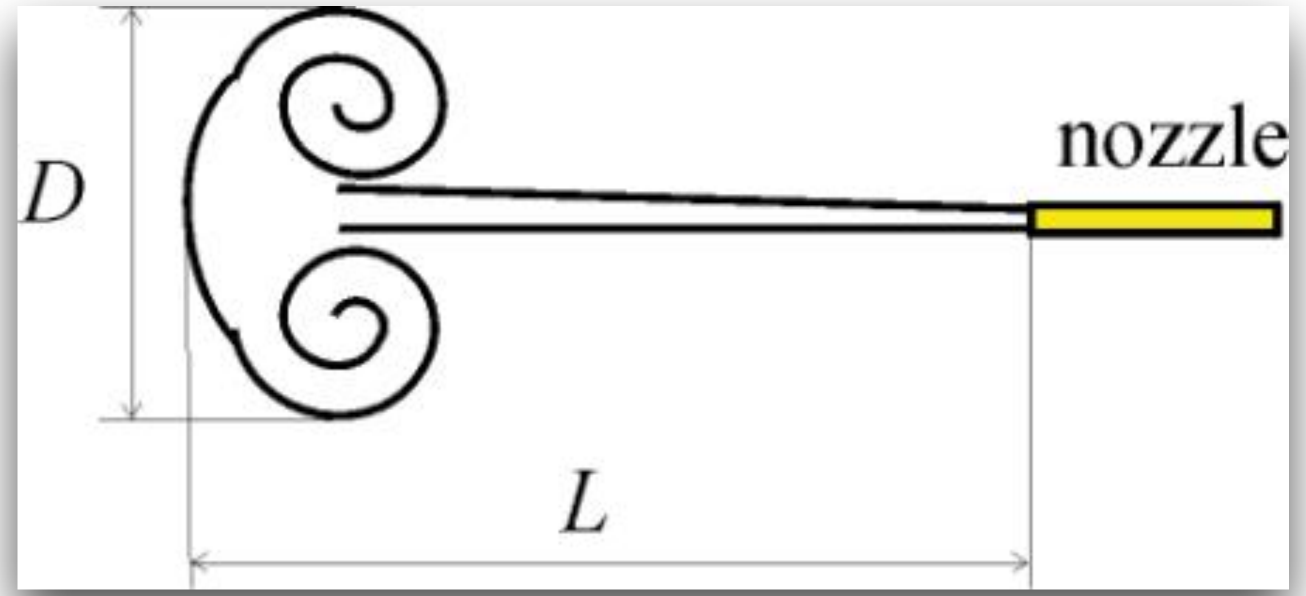
Vortex Dipole

A point vortex dipole consists of two vortices with equal strength but opposite circulation placed close together, forming a dipole.

- Vortices induce motion in each other, making the dipole move as a pair.
- The dipole travels in a straight line perpendicular to the line connecting the two vortices.



Vortex dipole formed by cold water (dark) in the central part of the sea.



Sketch of vortex dipole

Gröbli's Ancient Coordinate System

Specielle Probleme

über die

Bewegung geradliniger paralleler Wirbelfäden.

Inaugural-Dissertation

zur

Erlangung der Doctorwürde

der hohen philosophischen Facultät

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vorgelegt von

WALTER GRÖBLI,

von Oberutzwil, Canton St. Gallen.

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Druck von Zürcher und Furrer.

1877.

The evolution equations derived by Gröbli

$$\frac{d}{dt}\begin{pmatrix}\ell_{23}^2\\ \ell_{31}^2\\ \ell_{12}^2\end{pmatrix} = 4\sigma A \begin{pmatrix}\Gamma_1\left(\ell_{12}^{-2}-\ell_{31}^{-2}\right)\\ \Gamma_2\left(\ell_{23}^{-2}-\ell_{12}^{-2}\right)\\ \Gamma_3\left(\ell_{31}^{-2}-\ell_{23}^{-2}\right)\end{pmatrix}.$$

ℓ_{ij} be the distance between vortices i and j

A be the area of the triangle formed by the vortices

$\sigma = \pm 1$ indicate the orientation of the triangle spanned by the three vortices.

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- **Synge (1949)** Trilinear coordinates.
- **Aref** (1979) Rederivation of Gröbli's solution, and use of trilinear coordinates to interpret the results.

Trilinear Coordinate System

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For $L \neq 0$, Aref defines new variables

$$b_1 = \frac{l_{23}^2}{\Gamma_1 L}, \quad b_2 = \frac{l_{31}^2}{\Gamma_2 L}, \quad b_3 = \frac{l_{12}^2}{\Gamma_3 L},$$

such that $b_1 + b_2 + b_3 = 3$.

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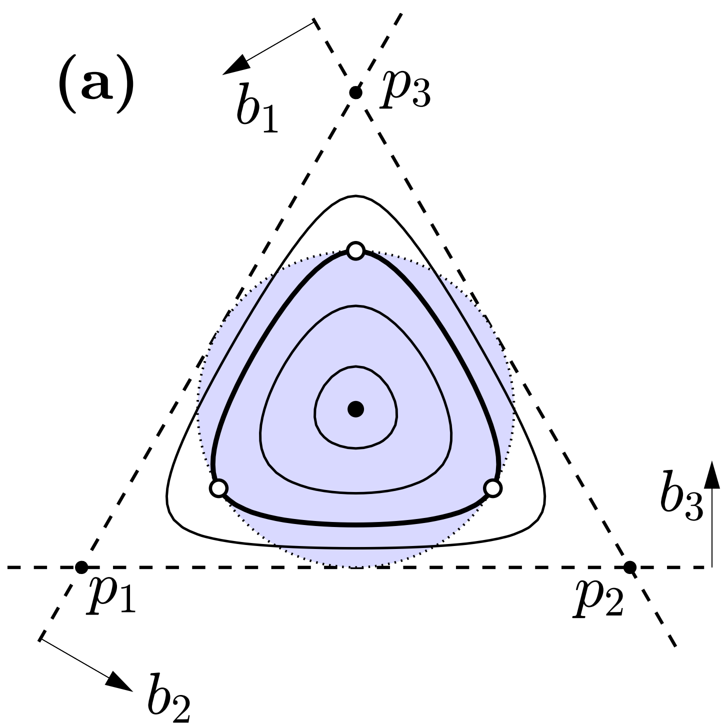

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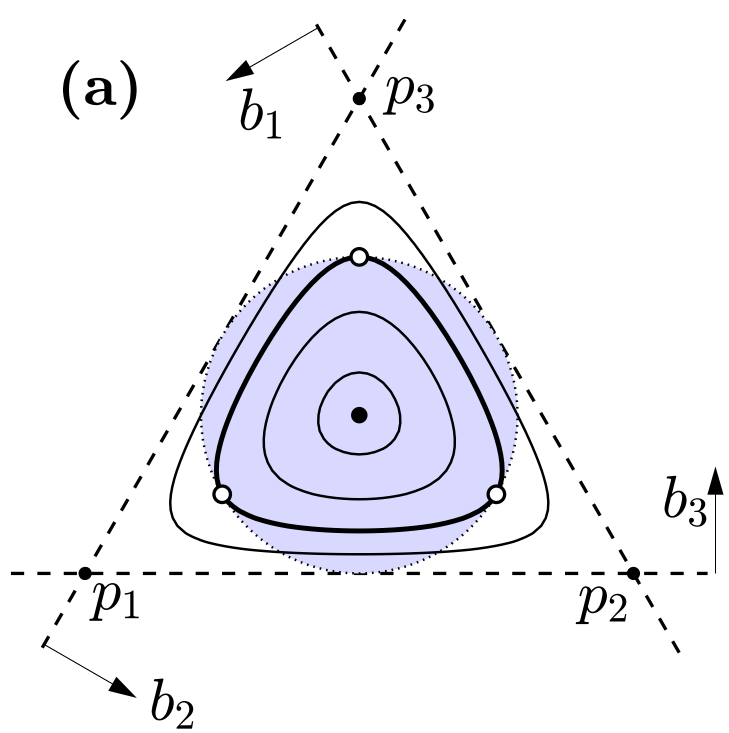

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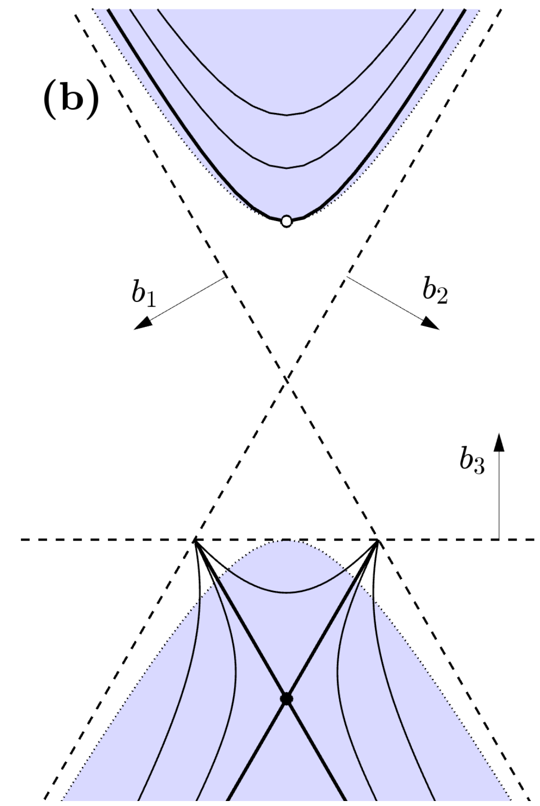

Vortices of circulation $(1, 1, -1)$

Limitations of previous coordinates system

- Not all triples satisfies the **triangular inequality constraint.**
- Introduces **coordinate singularities** at all collinear configuration
- The **linearization fails** at the boundaries due to collinear singularities.

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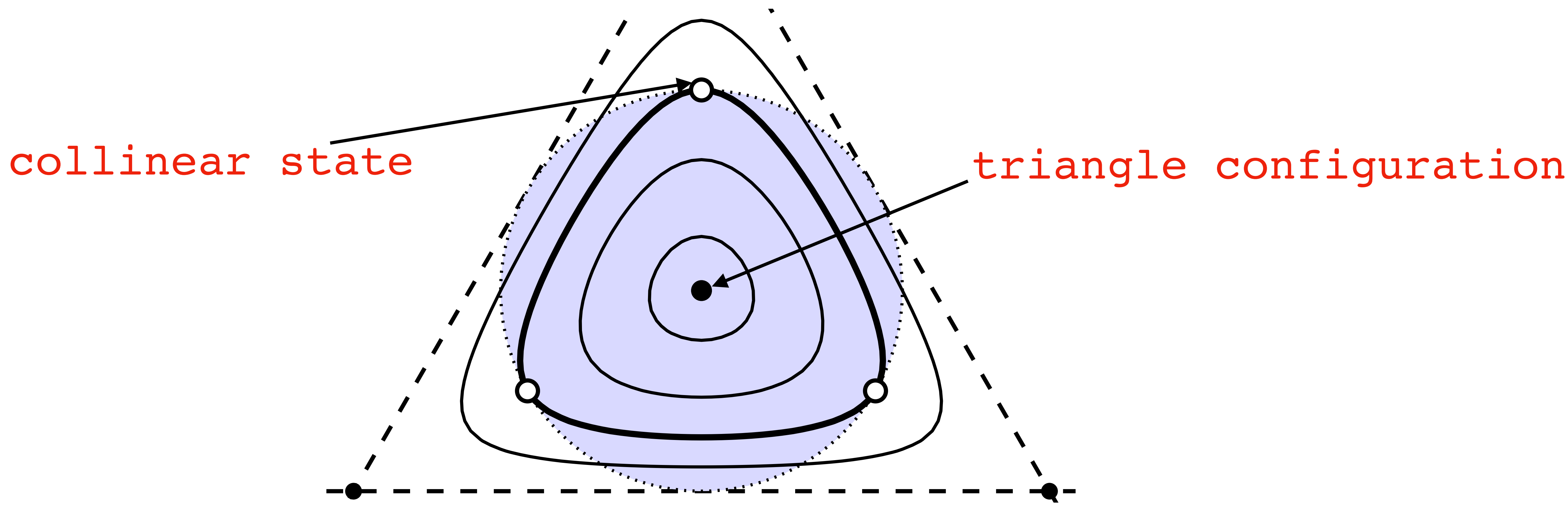

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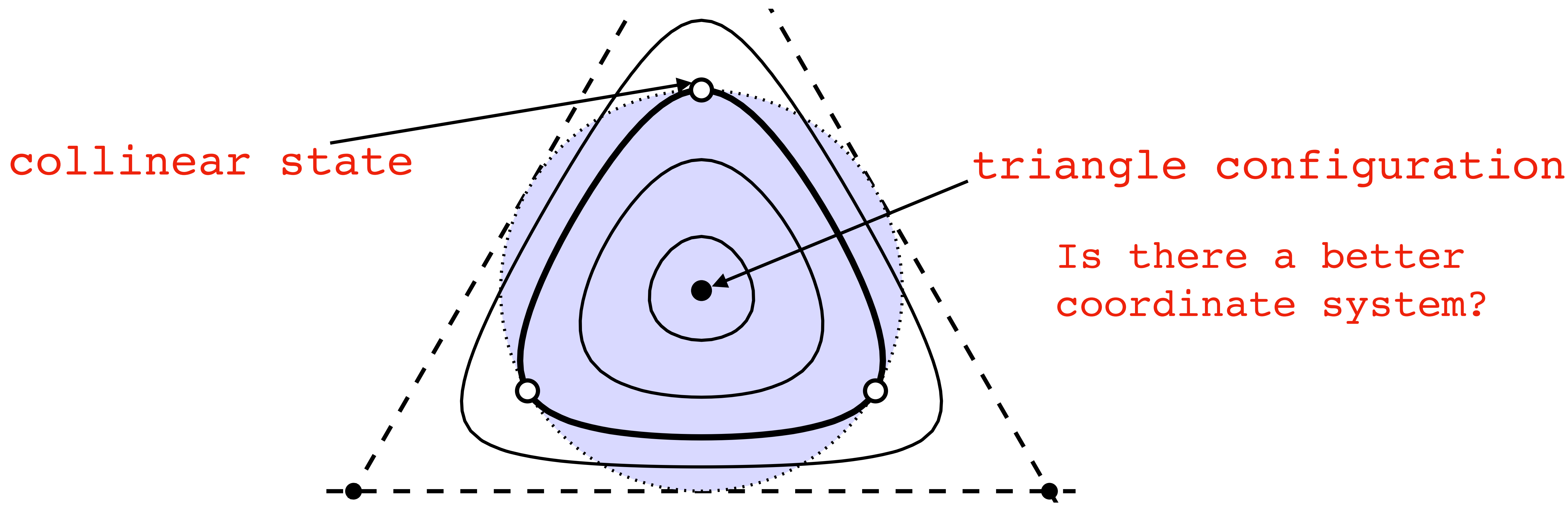

Is there a better coordinate system?

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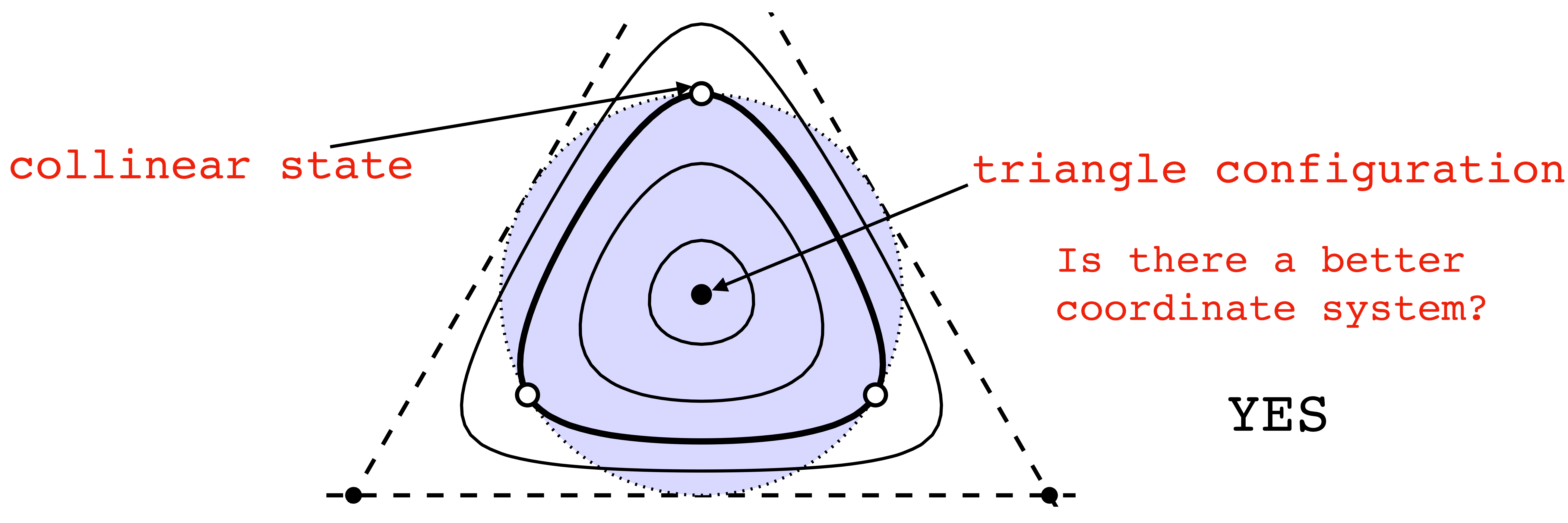

Jacobi coordinates

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Jacobi Coordinates (classical)

The first transformation

$$\tilde{\mathbf{r}}_1 = \mathbf{r}_1 - \mathbf{r}_2; \qquad \tilde{\Gamma}_1 = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2};$$

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The second transformation

$$R_1 = \tilde{r}_1; \qquad \kappa_1 = \tilde{\Gamma}_1;$$

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Transformed coordinates in terms of the physical coordinates:

$$R_1 = r_1 - r_2; \qquad \kappa_1 = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2};$$

$$R_2 = \frac{\Gamma_1 r_1 + \Gamma_2 r_2}{\Gamma_1 + \Gamma_2} - r_3; \qquad \kappa_2 = \frac{(\Gamma_1 + \Gamma_2)\Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3};$$

$$R_3 = \Gamma_1 r_1 + \Gamma_2 r_2 + \Gamma_3 r_3; \qquad \kappa_3 = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where $\Gamma_1 + \Gamma_2 + \Gamma_3 \neq 0$

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where $\Gamma_1 + \Gamma_2 + \Gamma_3 \neq 0$

Jacobi transformation is applied recursively.

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- Generalization of Poisson Brackets

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- Considers a canonical triplet

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The system with coordinates (X, Y, Z), the evolution equations

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$\Longleftarrow$

The reduced system is free from **Coordinate Singularities**

Application to the Three-Vortex System with Arbitrary Circulation

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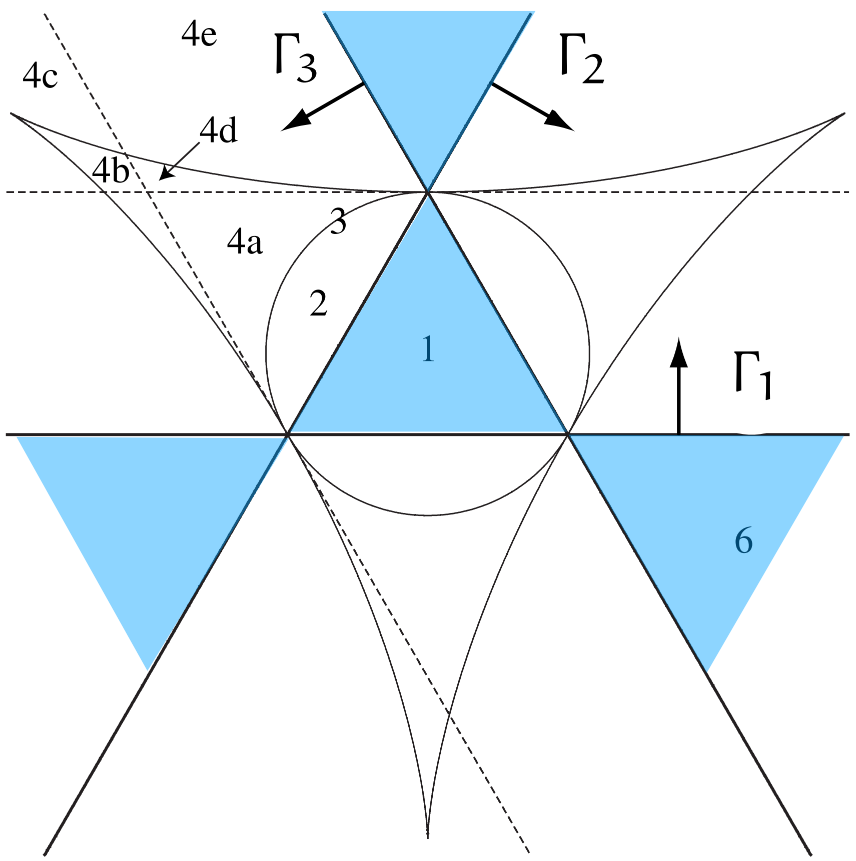
The three-vortex system conserve an energy when $\kappa_2 > 0$

$$H(X,Y,Z,\Theta) = -\frac{\Gamma_1\Gamma_2}{2}\log\left(\frac{Z+\Theta}{2\kappa_1}\right) - \frac{\Gamma_2\Gamma_3}{2}\log\left(\frac{\Theta-Z}{2\kappa_2} + \frac{\kappa_1(Z+\Theta)}{2\Gamma_2^2} - \frac{kX}{\Gamma_2}\right) - \frac{\Gamma_1\Gamma_3}{2}\log\left(\frac{\Theta-Z}{2\kappa_2} + \frac{\kappa_1(Z+\Theta)}{2\Gamma_1^2} + \frac{kX}{\Gamma_1}\right)$$

where $k^2 = \frac{\kappa_1}{\kappa_2}$.

$$\Theta^2 = Z^2 + X^2 + Y^2$$



Barycentric coordinates for circulation space when $\Gamma_1 + \Gamma_2 + \Gamma_3 = 1$

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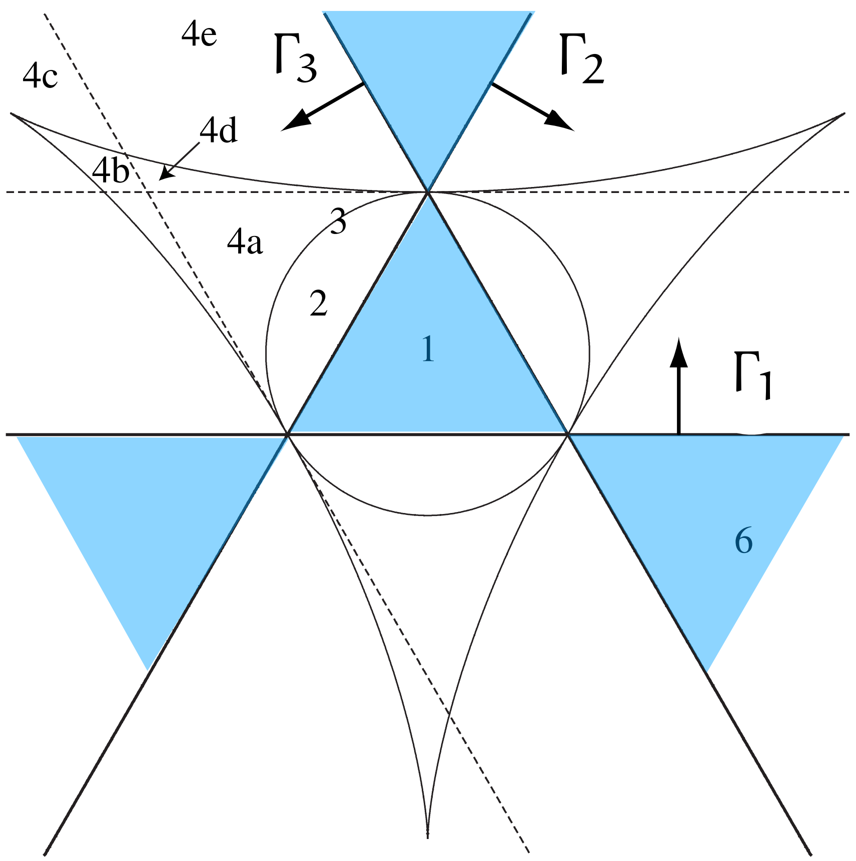

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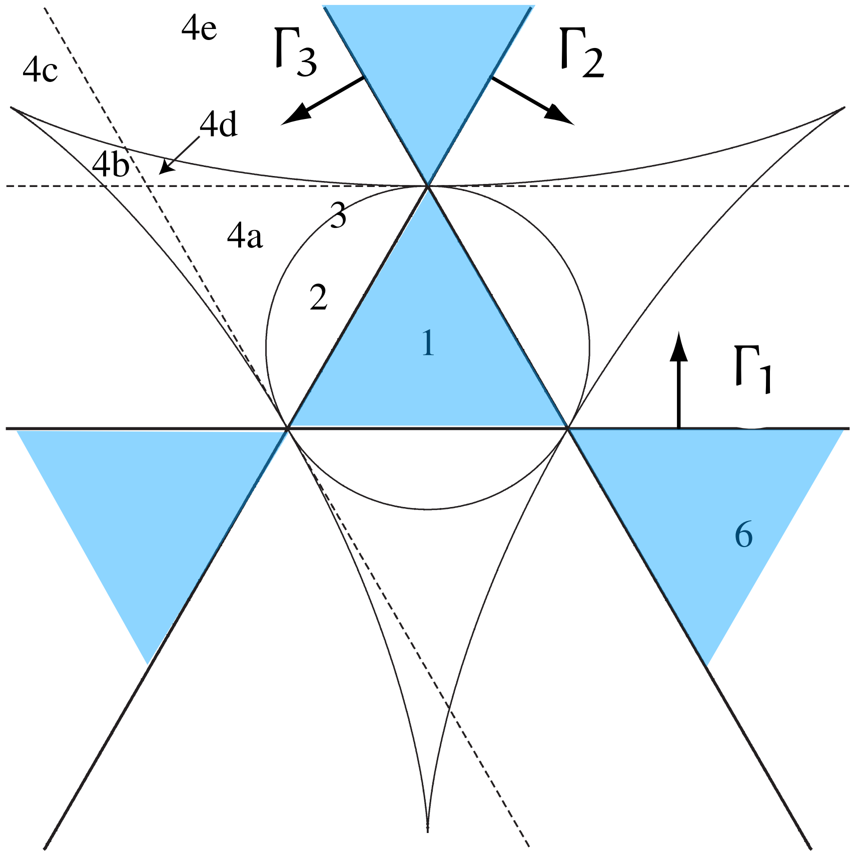

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Dynamics based of the sign of κ_2 :

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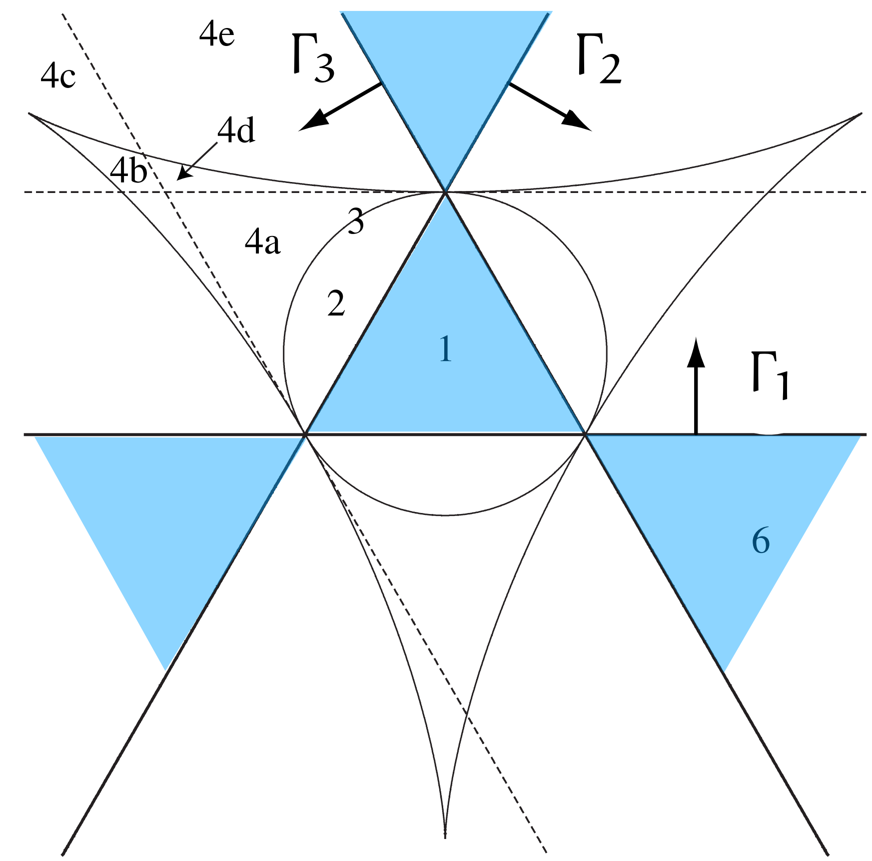

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Dynamics based of the sign of κ_2 :

$\kappa_2 > 0$

(X, Y, Z, Θ) represents a sphere

Application to the Three-Vortex System with Arbitrary Circulation

The three-vortex system conserve an energy when $\kappa_2 > 0$

$$H(X,Y,Z,\Theta) = -\frac{\Gamma_1\Gamma_2}{2}\log\left(\frac{Z+\Theta}{2\kappa_1}\right) - \frac{\Gamma_2\Gamma_3}{2}\log\left(\frac{\Theta-Z}{2\kappa_2} + \frac{\kappa_1(Z+\Theta)}{2\Gamma_2^2} - \frac{kX}{\Gamma_2}\right) - \frac{\Gamma_1\Gamma_3}{2}\log\left(\frac{\Theta-Z}{2\kappa_2} + \frac{\kappa_1(Z+\Theta)}{2\Gamma_1^2} + \frac{kX}{\Gamma_1}\right)$$

where $k^2 = \frac{\kappa_1}{\kappa_2}$.

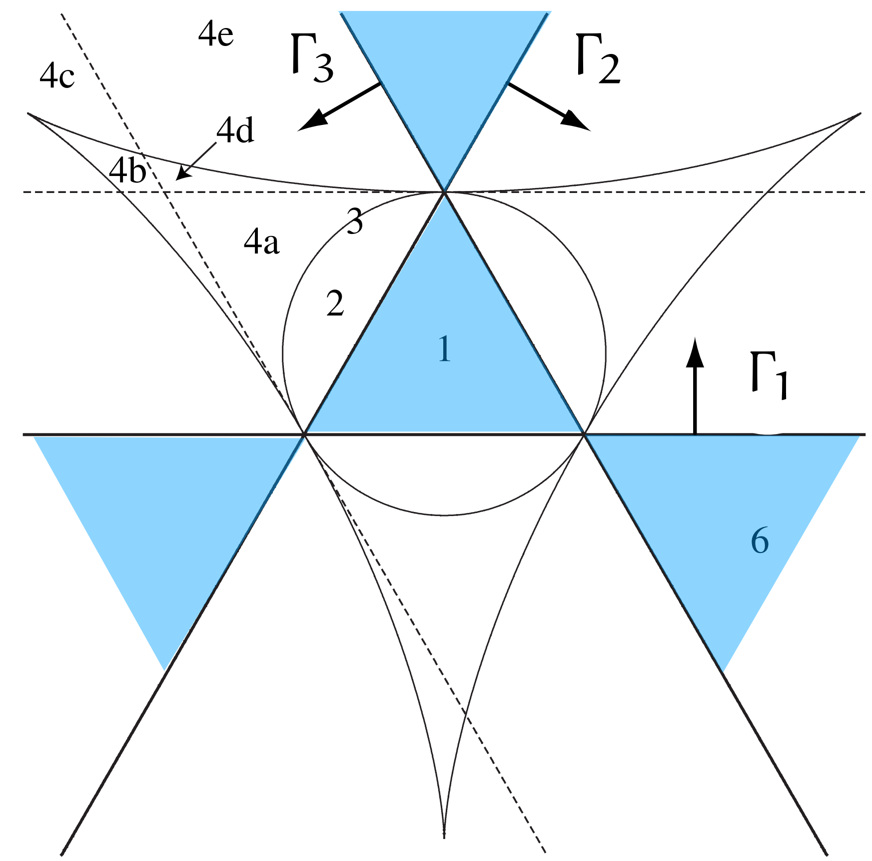
$$\Theta^2 = Z^2 + X^2 + Y^2$$

The three-vortex system conserve an energy when $\kappa_2 < 0$

$$H(X,Y,Z,\Theta) = -\frac{\Gamma_1\Gamma_2}{2}\log\left(\frac{Z+\Theta}{2\kappa_1}\right) - \frac{\Gamma_2\Gamma_3}{2}\log\left(\frac{Z-\Theta}{2\kappa_2} + \frac{\kappa_1(Z+\Theta)}{2\Gamma_2^2} - \frac{lX}{\Gamma_2}\right) - \frac{\Gamma_1\Gamma_3}{2}\log\left(\frac{Z-\Theta}{2\kappa_2} + \frac{\kappa_1(Z+\Theta)}{2\Gamma_1^2} + \frac{lX}{\Gamma_1}\right),$$

where $l^2 = \frac{-\kappa_1}{\kappa_2}$.

$$\Theta^2 = Z^2 - X^2 - Y^2$$



Barycentric coordinates for circulation space when $\Gamma_1 + \Gamma_2 + \Gamma_3 = 1$

Dynamics based of the sign of κ_2 :

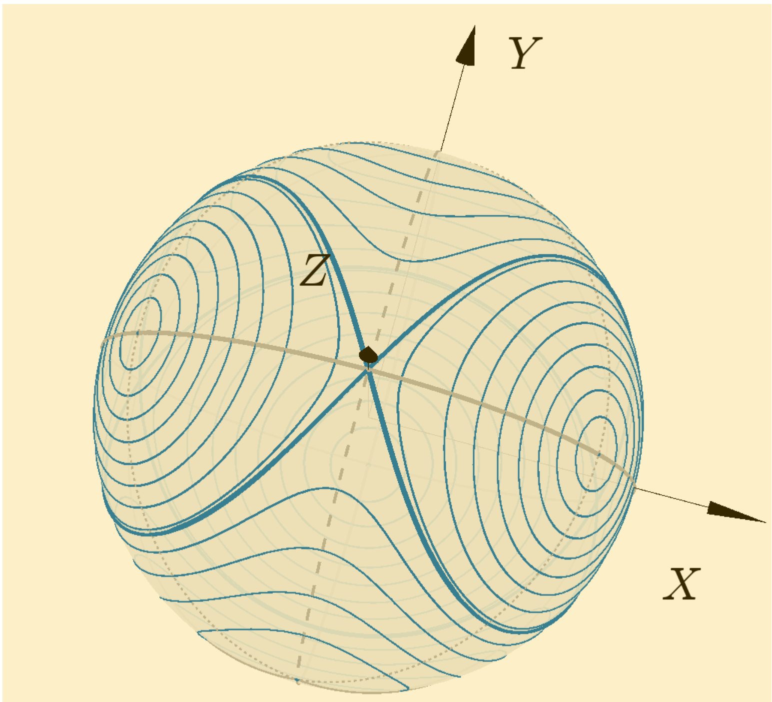
$\kappa_2 > 0$

(X, Y, Z, Θ) represents a sphere

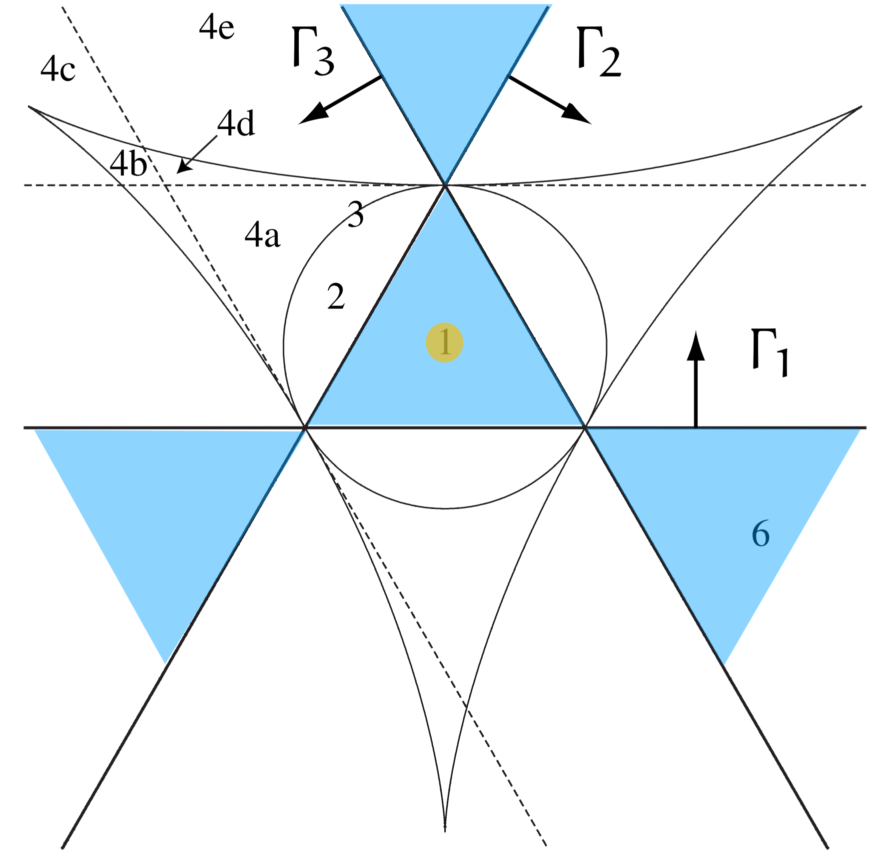
$\kappa_2 < 0$

(X, Y, Z, Θ) represents a upper sheet of a two-sheeted hyperboloid

$\kappa_2 > 0$ (Sphere)



$$(\Gamma_1, \Gamma_2, \Gamma_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$



Barycentric coordinates for circulation space when $\Gamma_1 + \Gamma_2 + \Gamma_3 = 1$

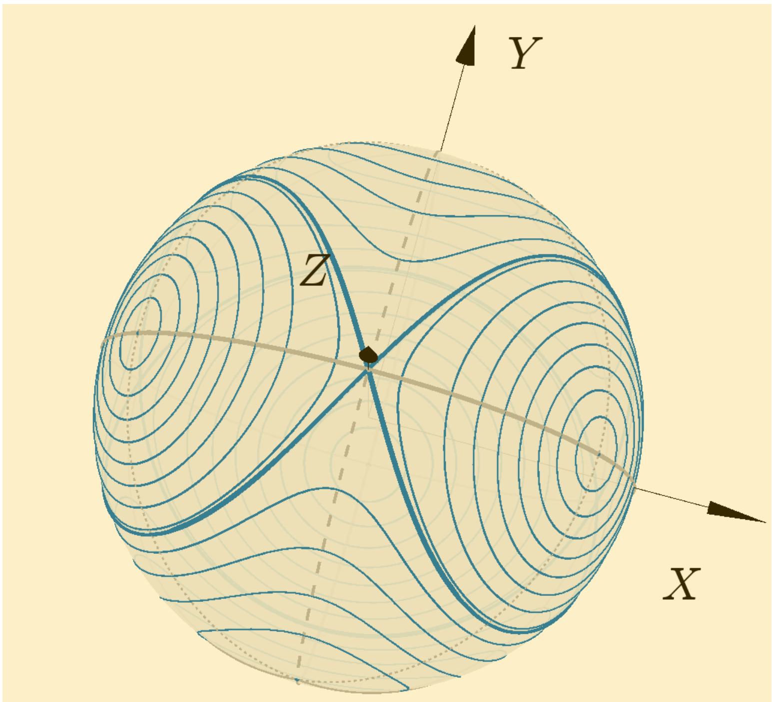
Fixed Point Analysis (Jacobian)

$$J(0,1,0) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad J(1,0,0) = \begin{pmatrix} 0 & 3 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Triangular Equilibrium (stable)

Collinear Equilibrium (saddle)

$\kappa_2 > 0$ (Sphere)



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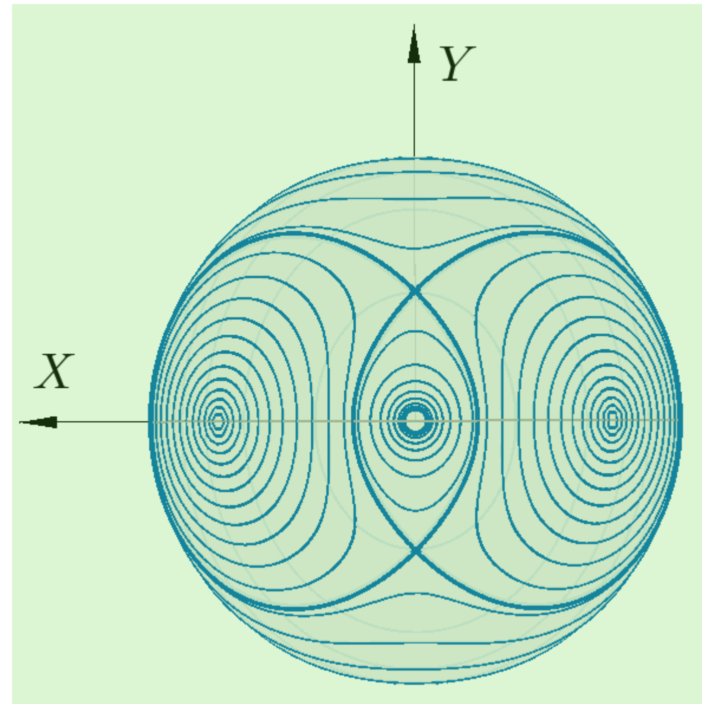
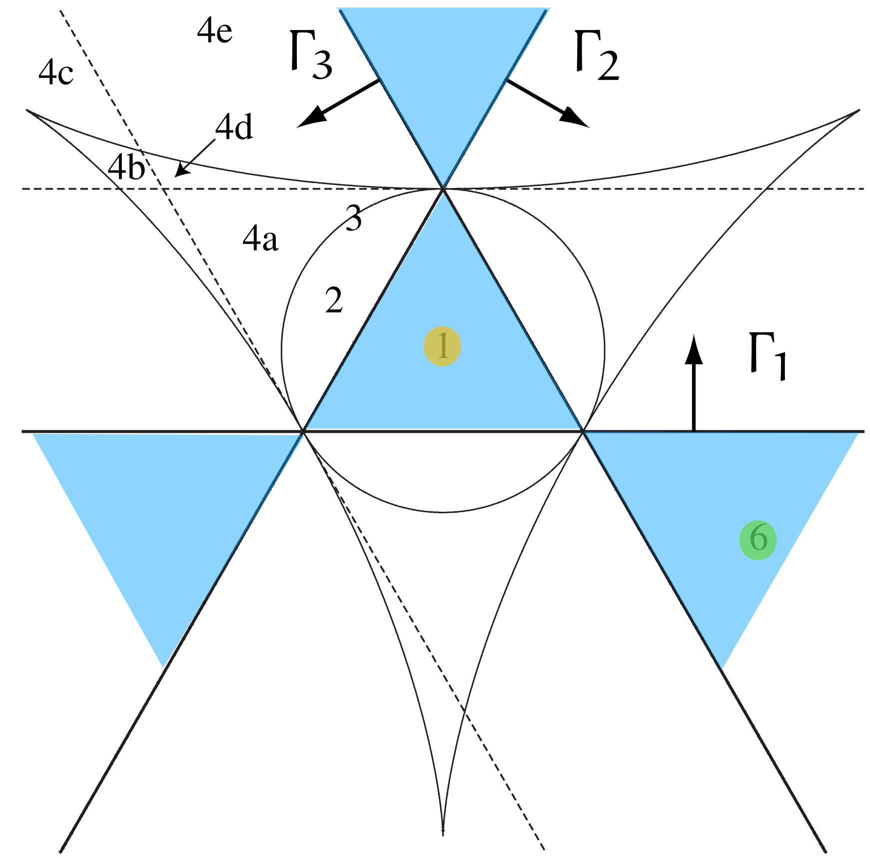


$(\Gamma_1, \Gamma_2, \Gamma_3) = (-2, -1, 4)$



Barycentric coordinates for circulation space when $\Gamma_1 + \Gamma_2 + \Gamma_3 = 1$

Fixed Point Analysis (Jacobian)

$(0, 0, 1)$ is a center (stable).

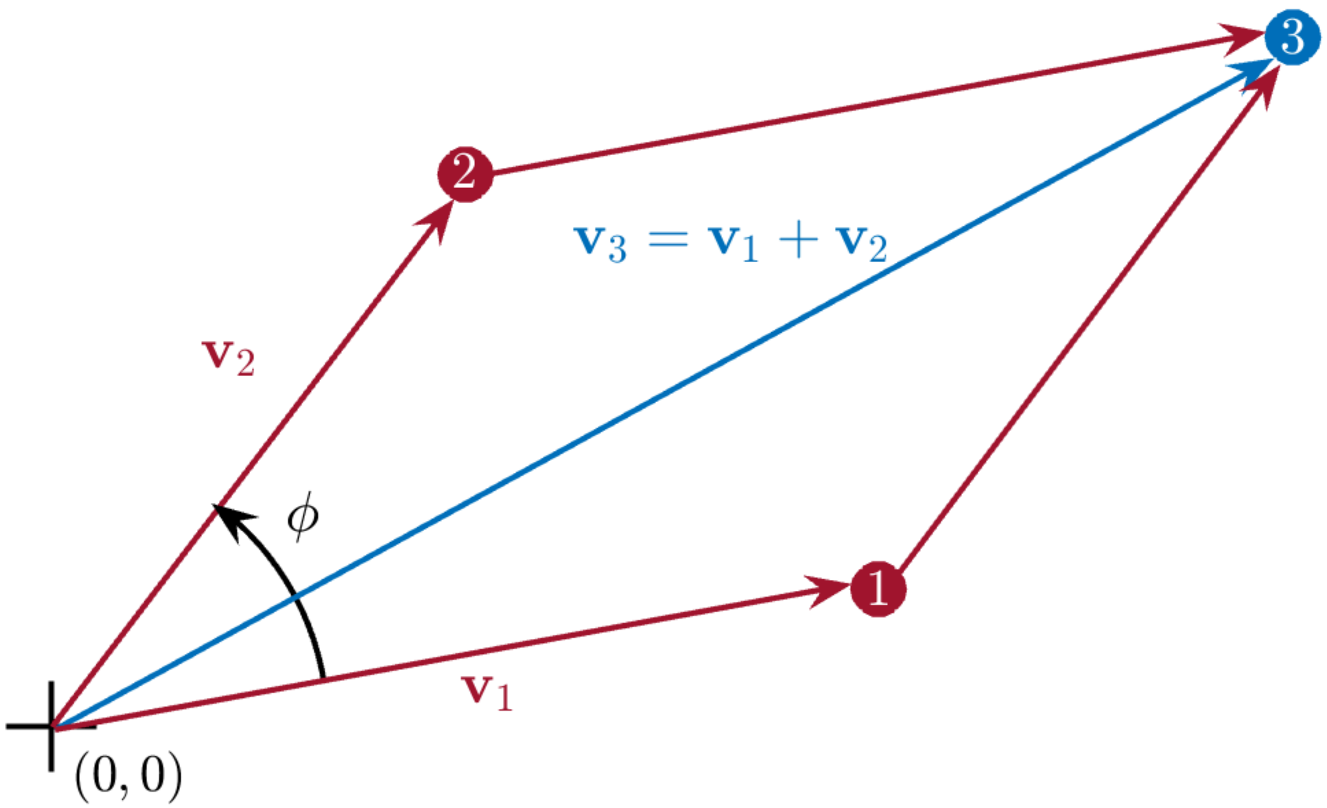
$$J(0,1,0) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad J(1,0,0) = \begin{pmatrix} 0 & 3 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\left(0, \pm\frac{3}{5}, \pm\frac{4}{5}\right)$ is a saddle.

Triangular Equilibrium (stable)

Collinear Equilibrium (saddle)

$\Gamma = (1, 1, -1) : \kappa_2 < 0$ (Hyperboloid)



New Coordinates

$$X = -\|v_1\|^2 + \|v_2\|^2;$$

$$Y = 2\,\|v_1\|\|v_2\| \sin\phi;$$

$$Z = \|v_1\|^2 + \|v_2\|^2;$$

$$\Theta = -2\,\|v_1\|\|v_2\| \cos\phi.$$

which satisfy:

$$\frac{dX}{dt} = \frac{-2Y}{Z+\Theta} + \frac{4ZY}{Z^2 - X^2};$$

$$\frac{dY}{dt} = \frac{2X}{Z+\Theta};$$

$$\frac{dZ}{dt} = \frac{4XY}{Z^2 - X^2},$$

$$\Theta^2 = Z^2 - X^2 - Y^2$$

(conserved quantity)

Phase Space when $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, -1)$

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The space is the upper sheet of a two-sheeted hyperbola.

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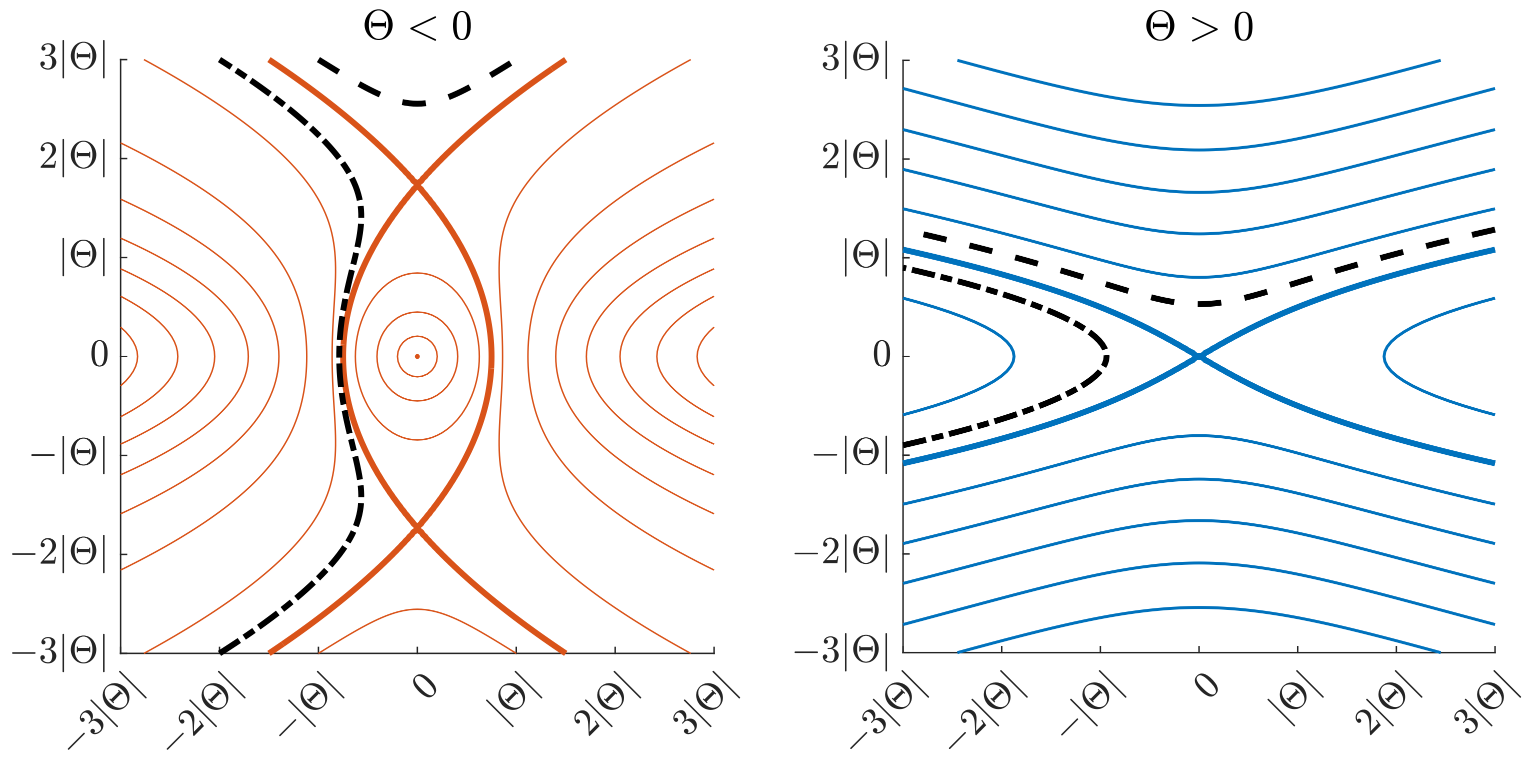

Phase Space when $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, -1)$

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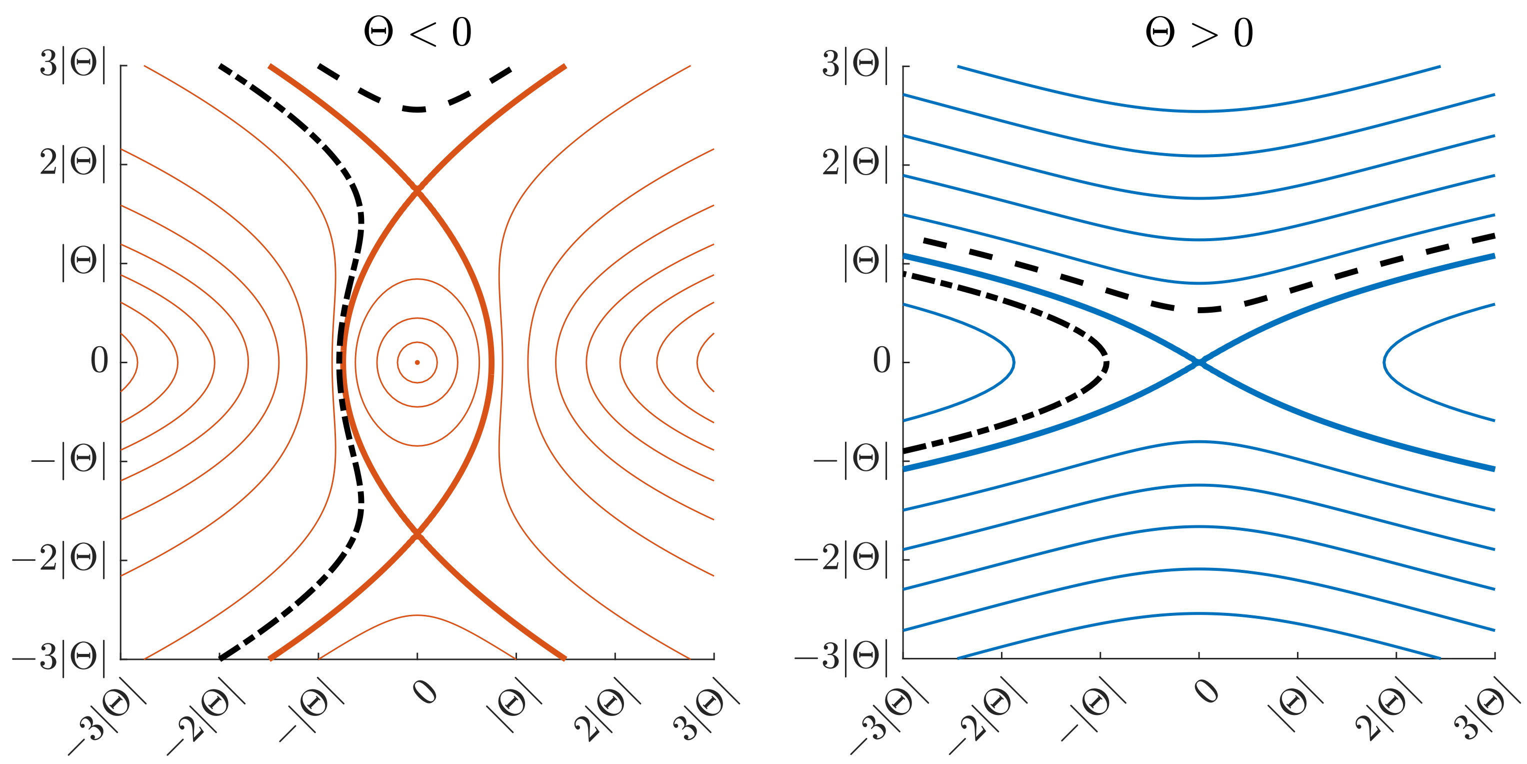
The level sets of the Hamiltonian are plotted in the phase planes



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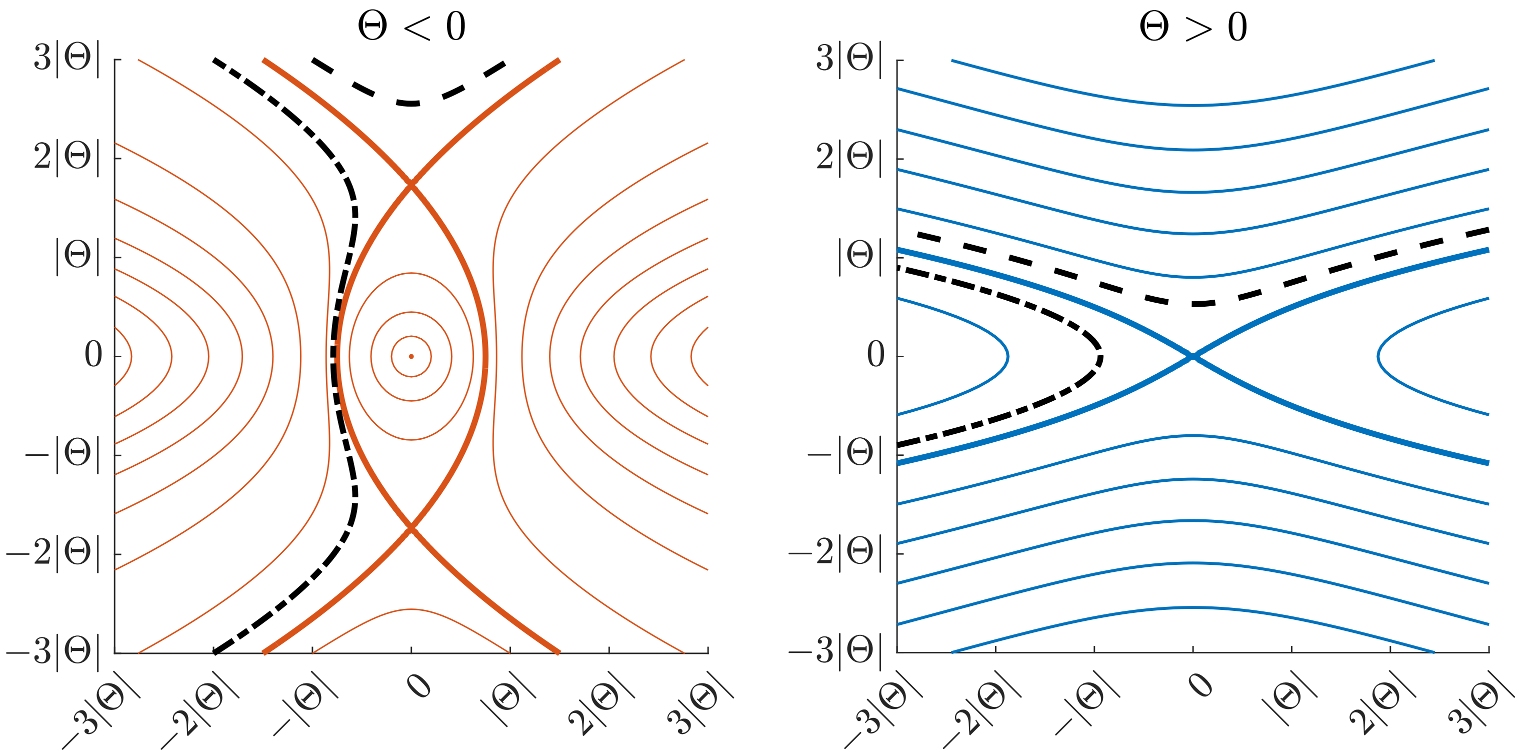

The level sets of the Hamiltonian are plotted in the phase planes

What do they represent in Aref's Phase Plane diagram?

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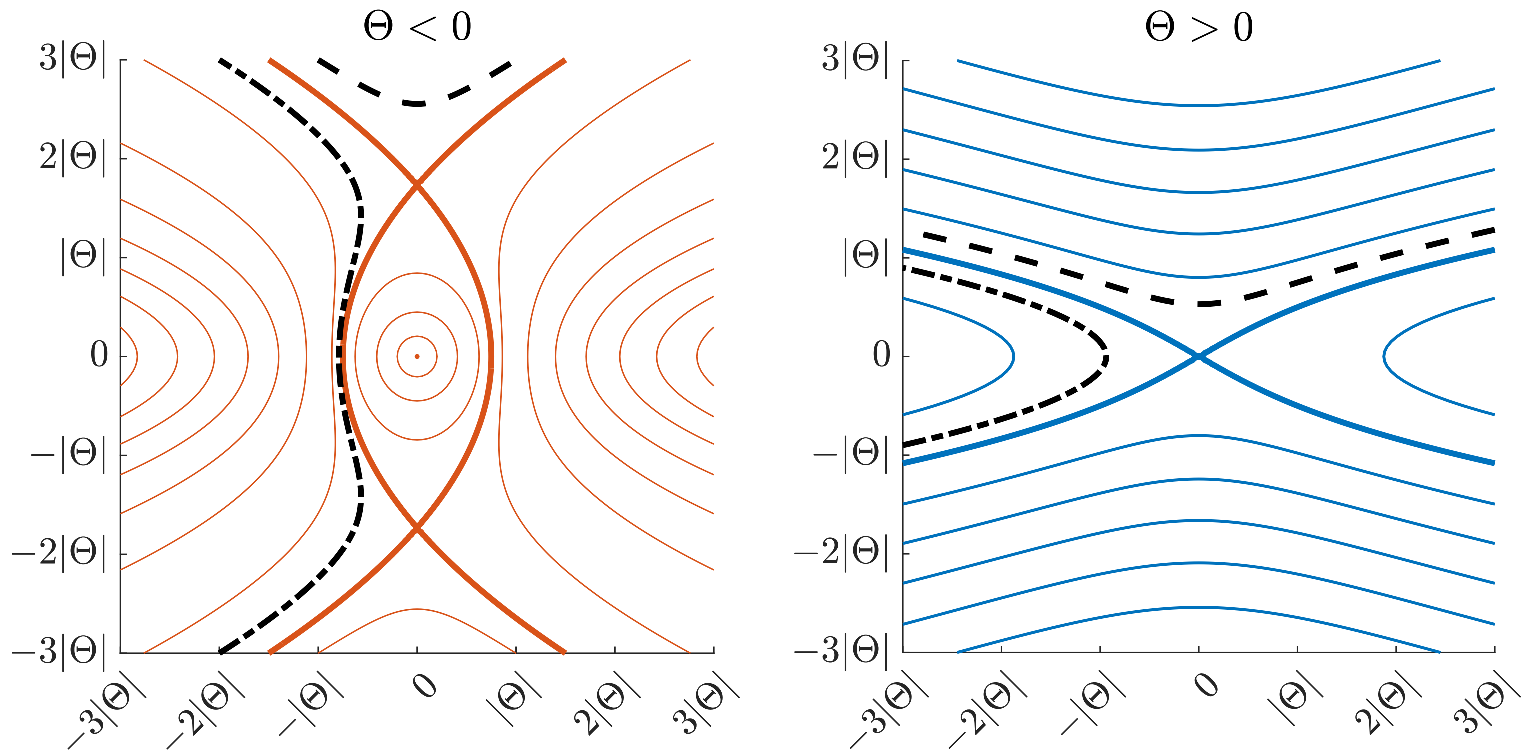

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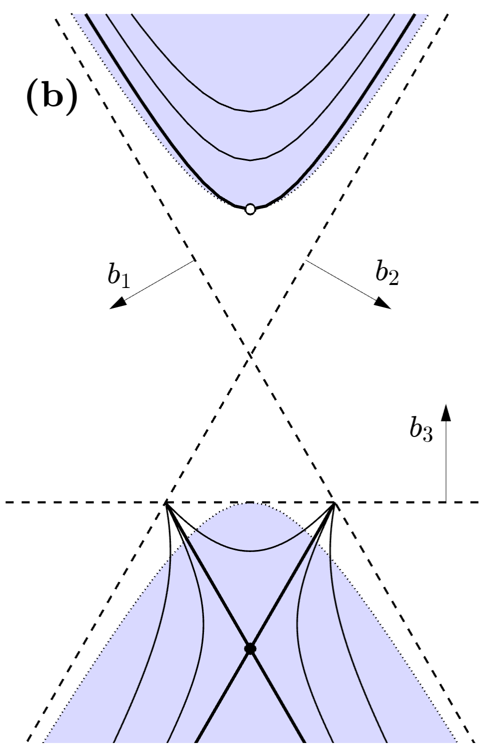

Vortices of circulation $(1, 1, -1)$

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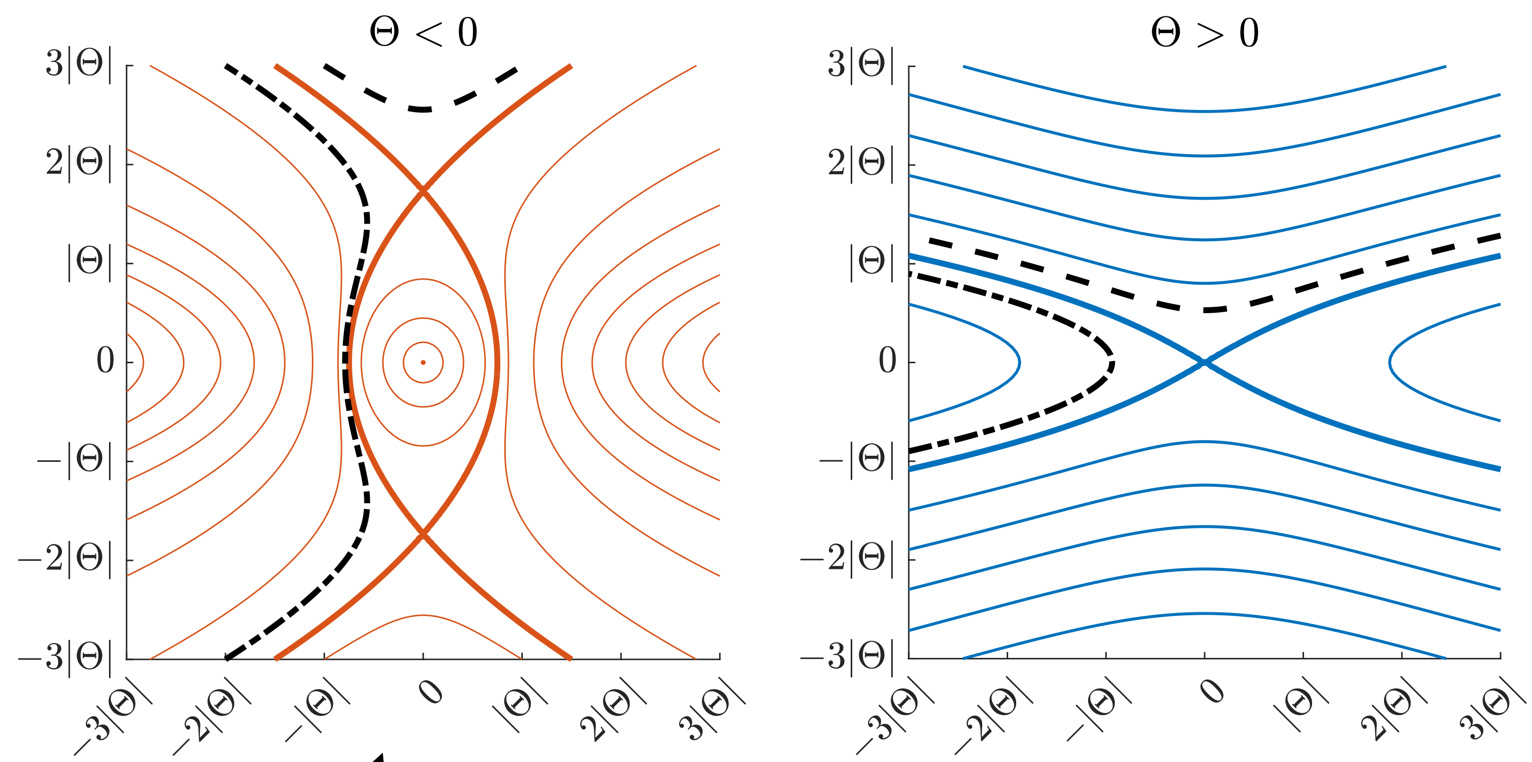

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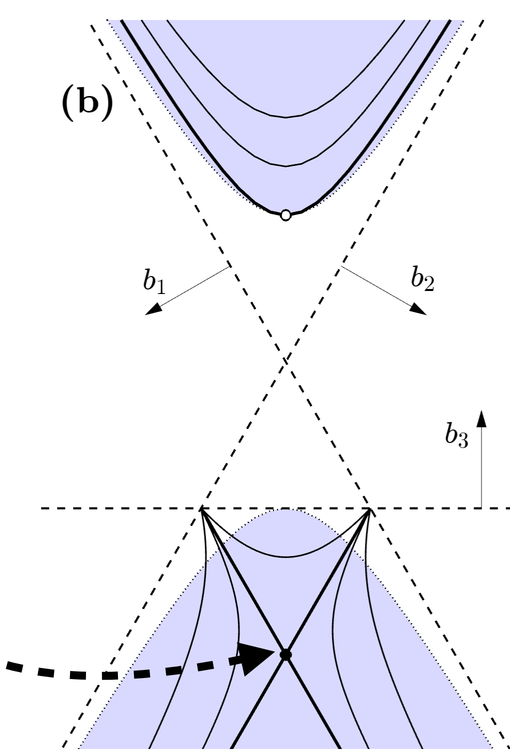
Triangular configuration where at the separatrix



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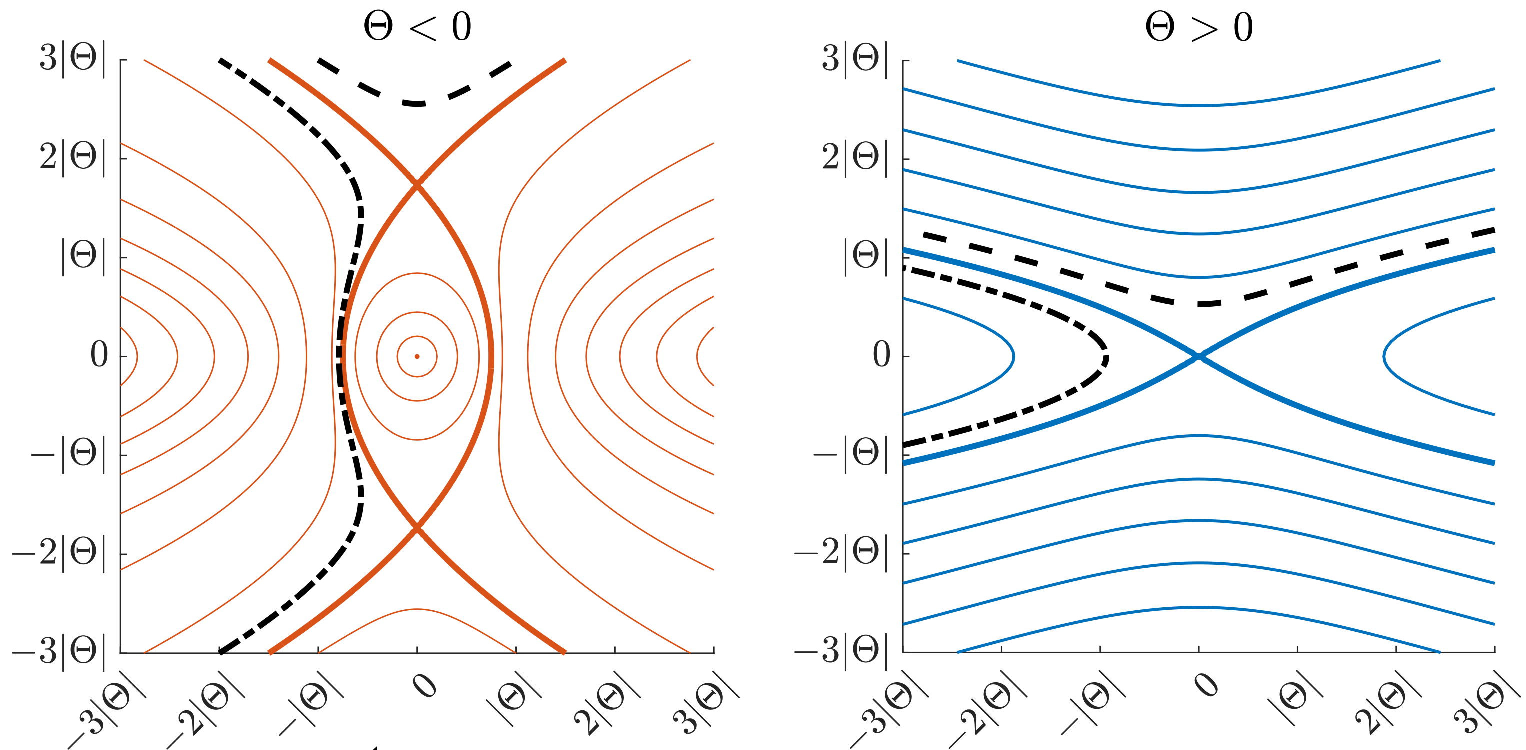

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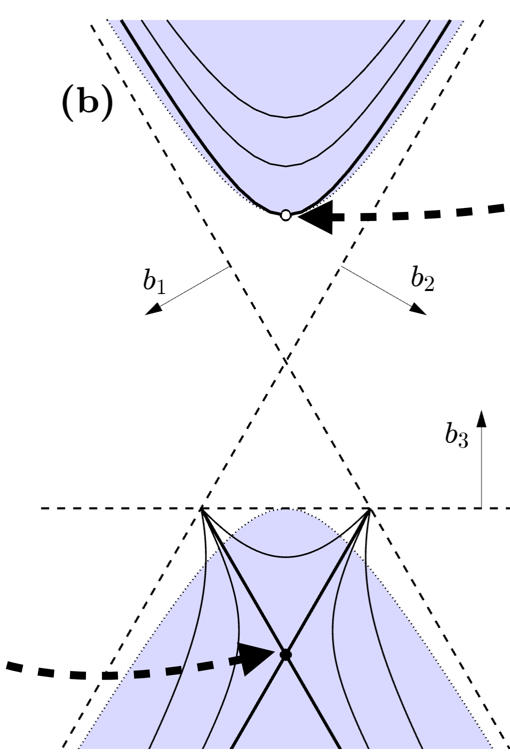

Collinear configuration at the origin

Triangular configuration where at the separatrix

Vortices of circulation $(1, 1, -1)$

Application to vortex dipole scattering for $\Gamma = (1, 1, -1)$

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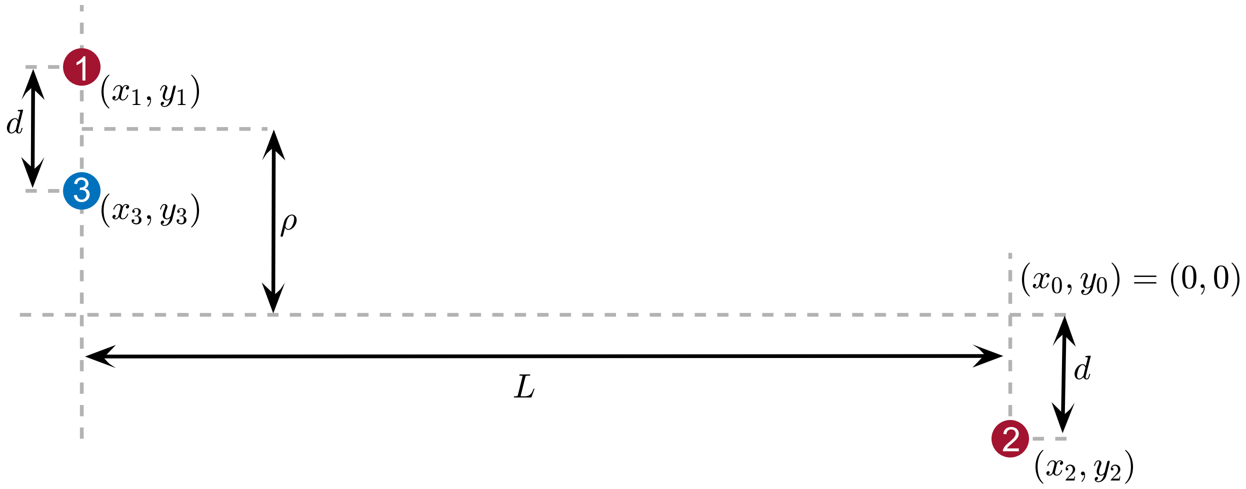
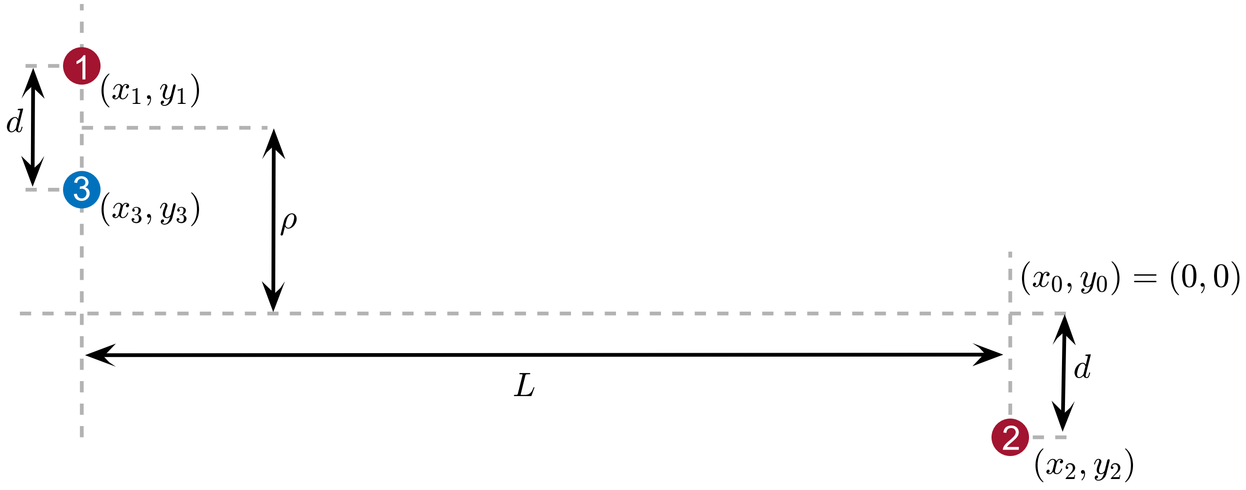


Setup

Application to vortex dipole scattering for $\Gamma = (1, 1, -1)$



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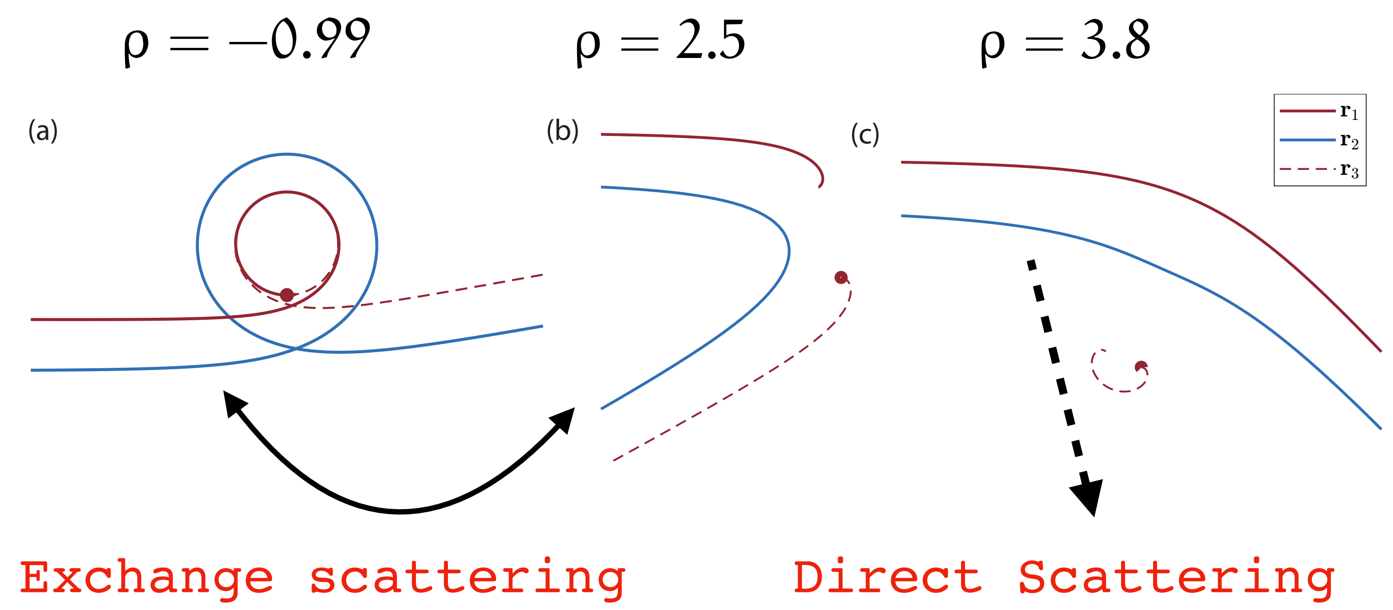

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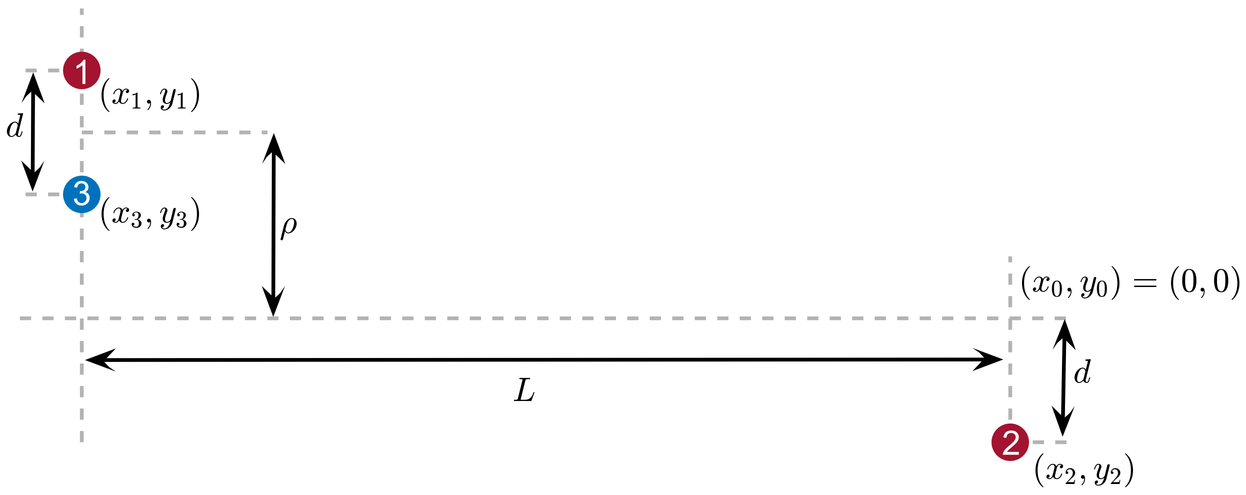
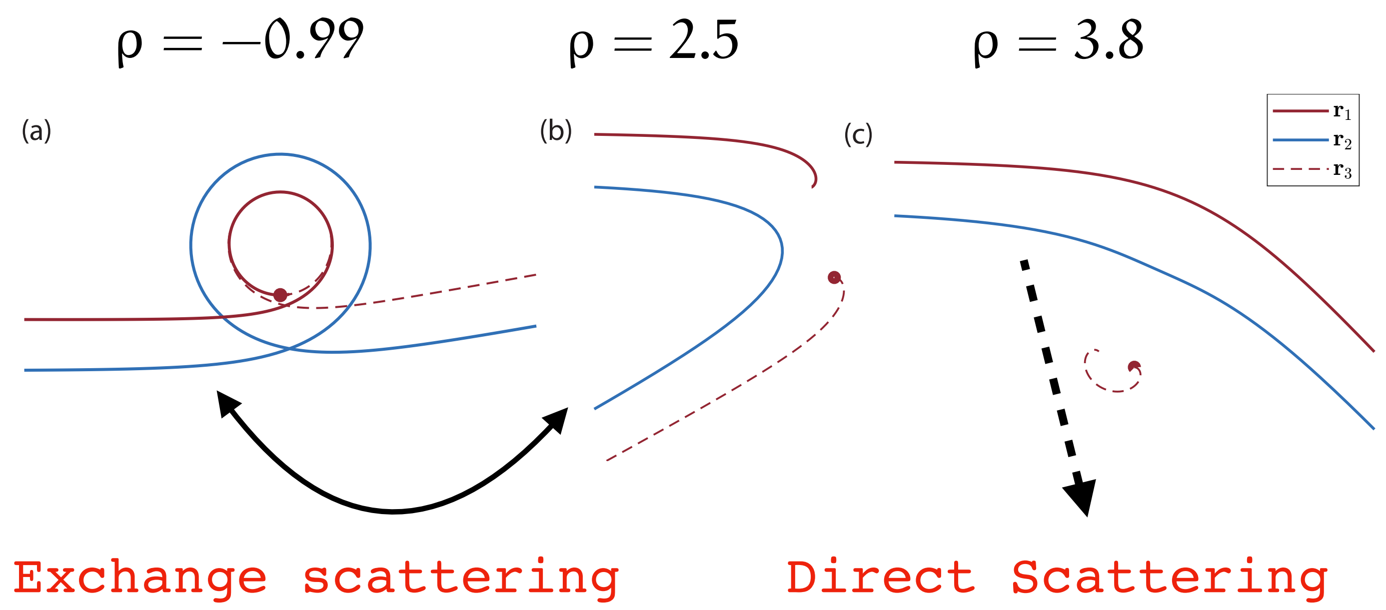
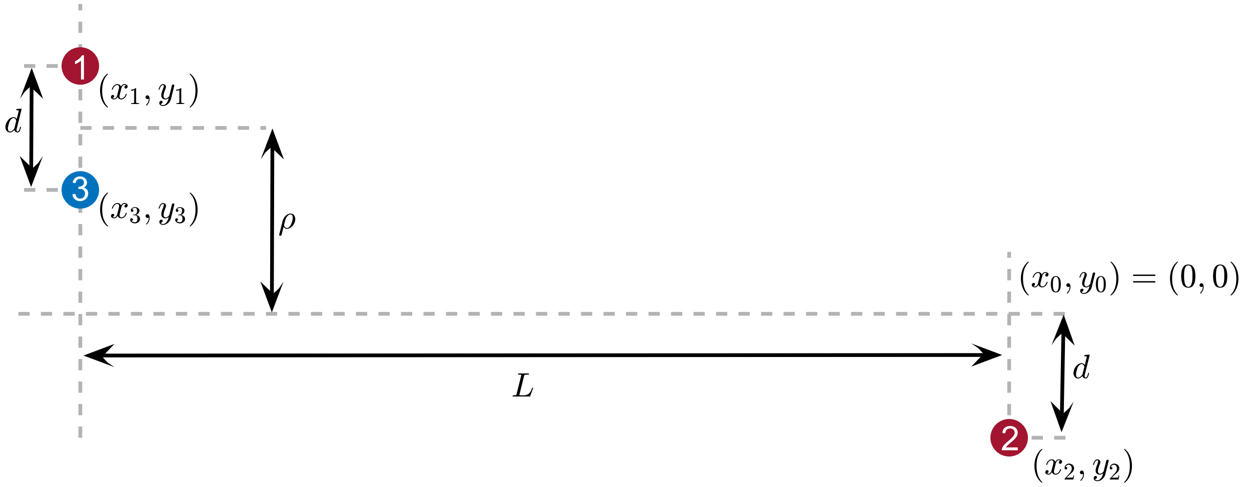


Setup

H and Θ are determined by the parameter ρ



Application to vortex dipole scattering for $\Gamma = (1, 1, -1)$



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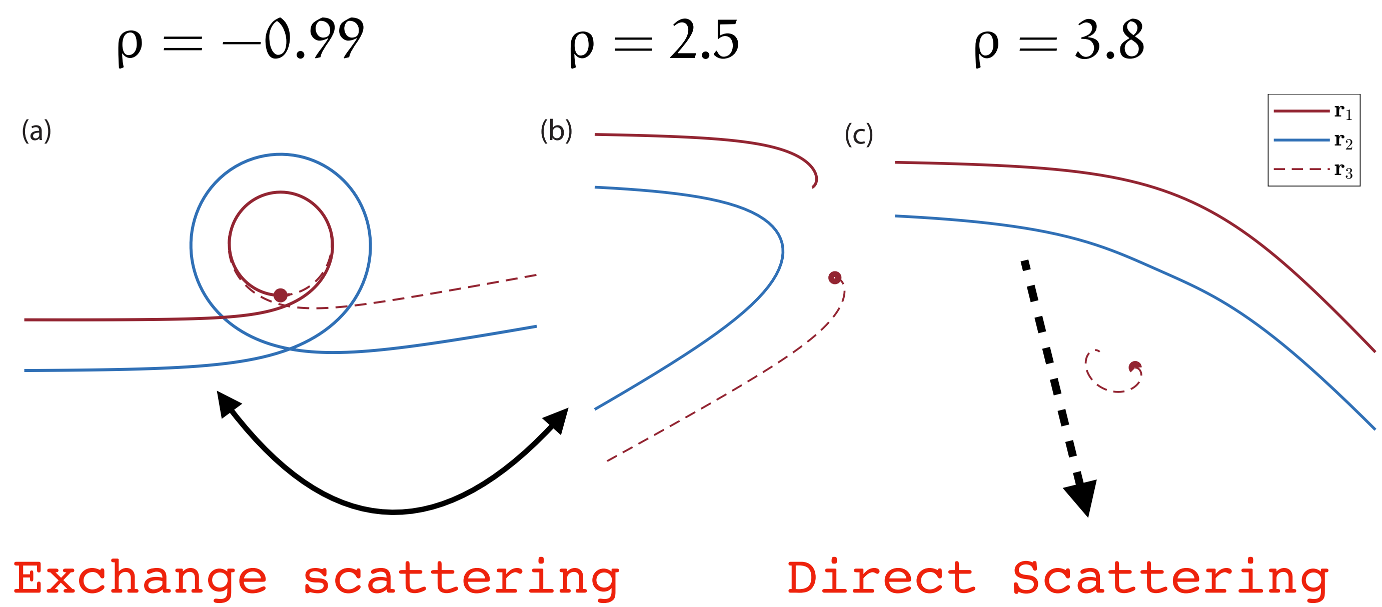
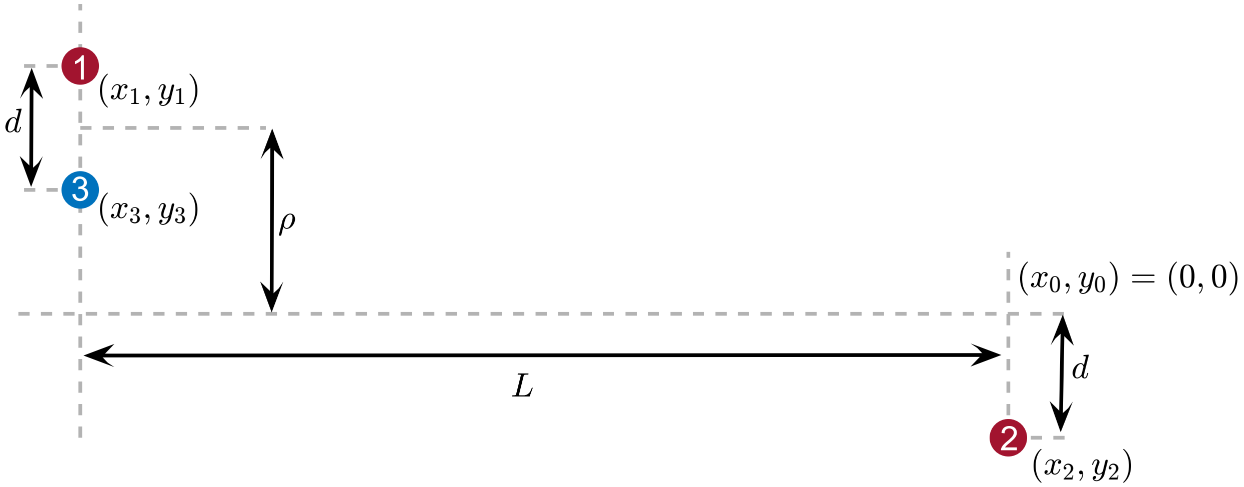


H and Θ are determined by the parameter ρ

At the critical values of ρ, the nature of the scattering changes

Application to vortex dipole scattering for $\Gamma = (1, 1, -1)$



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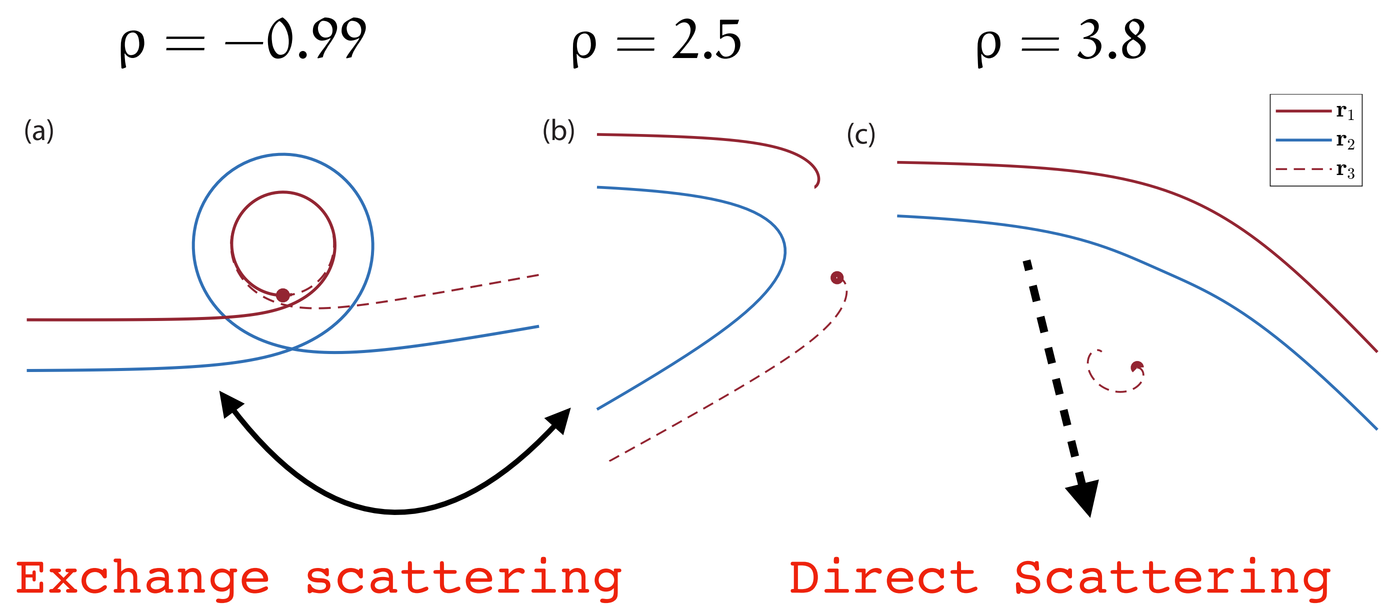

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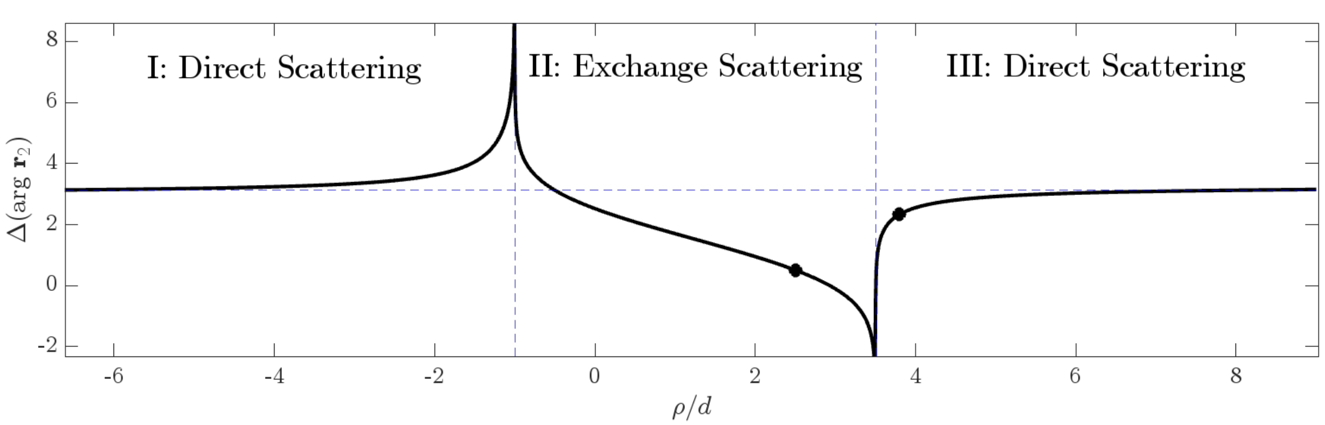

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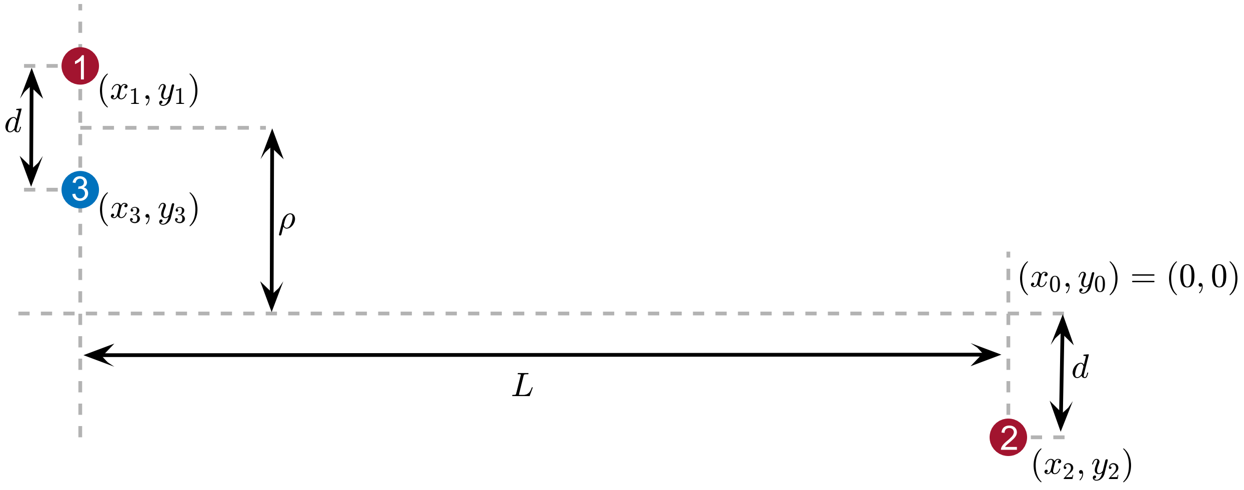

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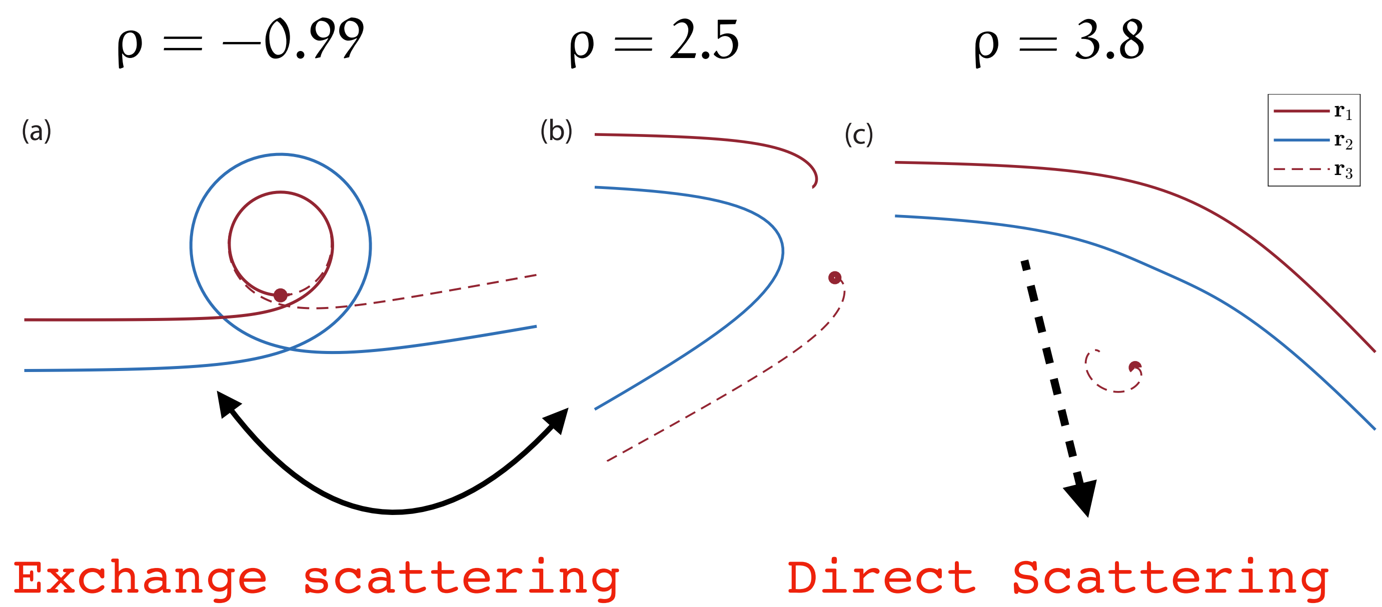

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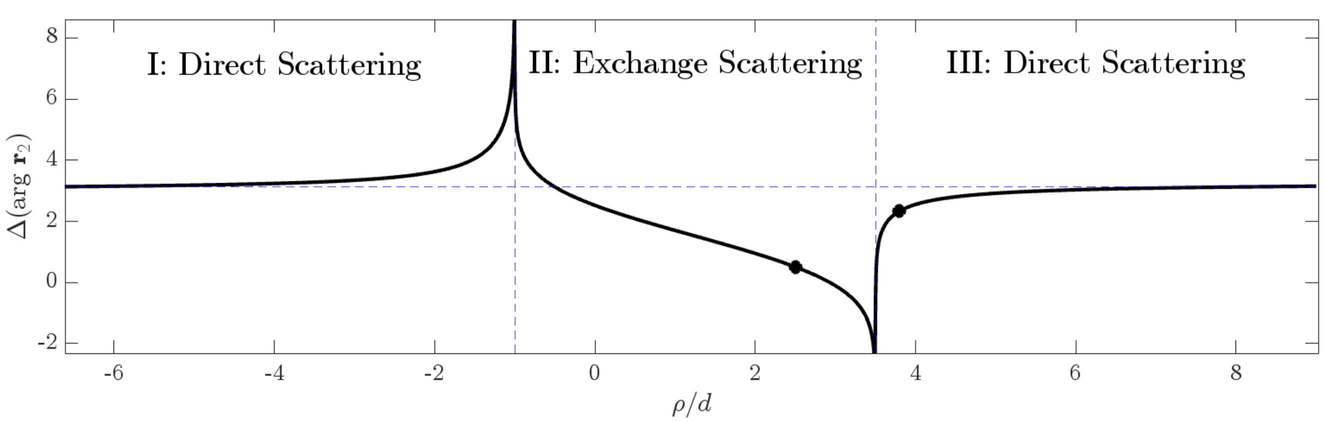

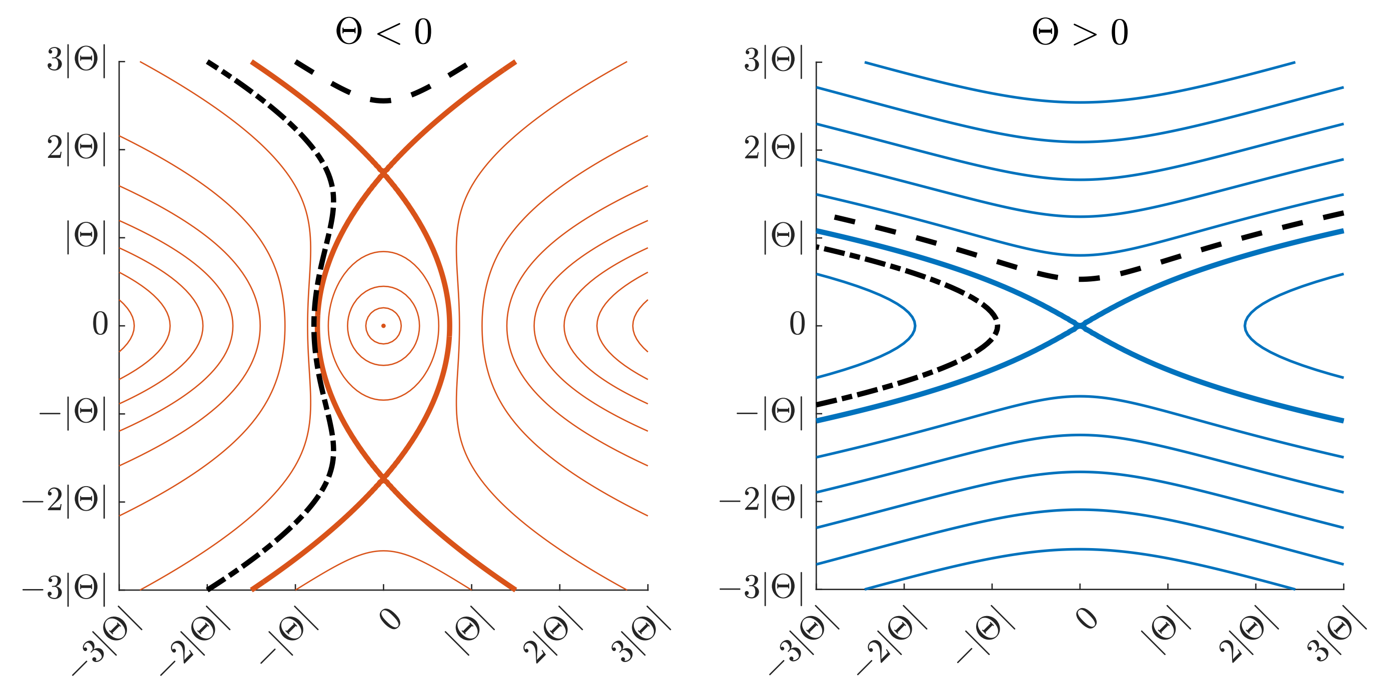

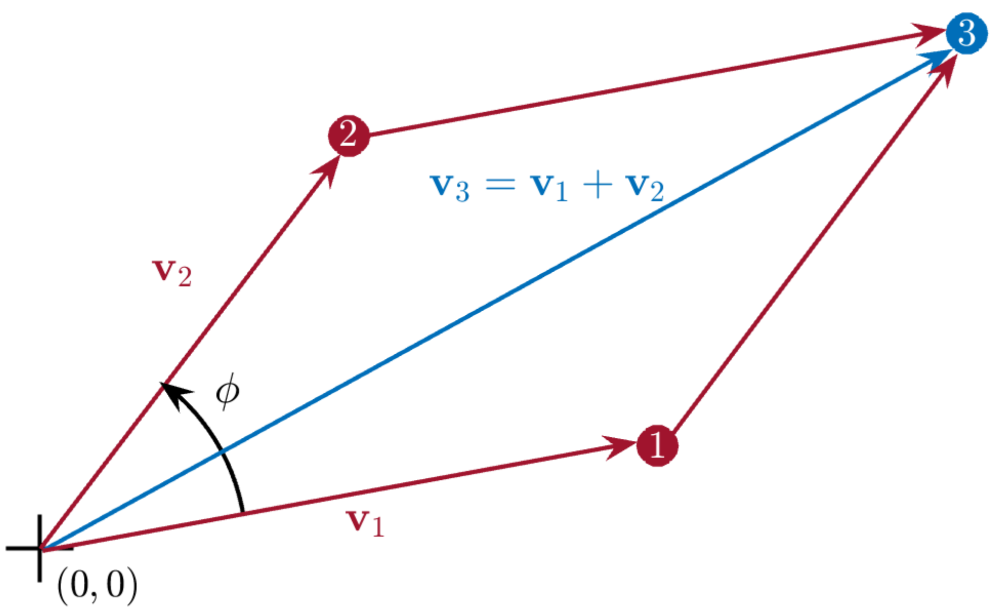

$$X = -\|v_1\|^2 + \|v_2\|^2;$$
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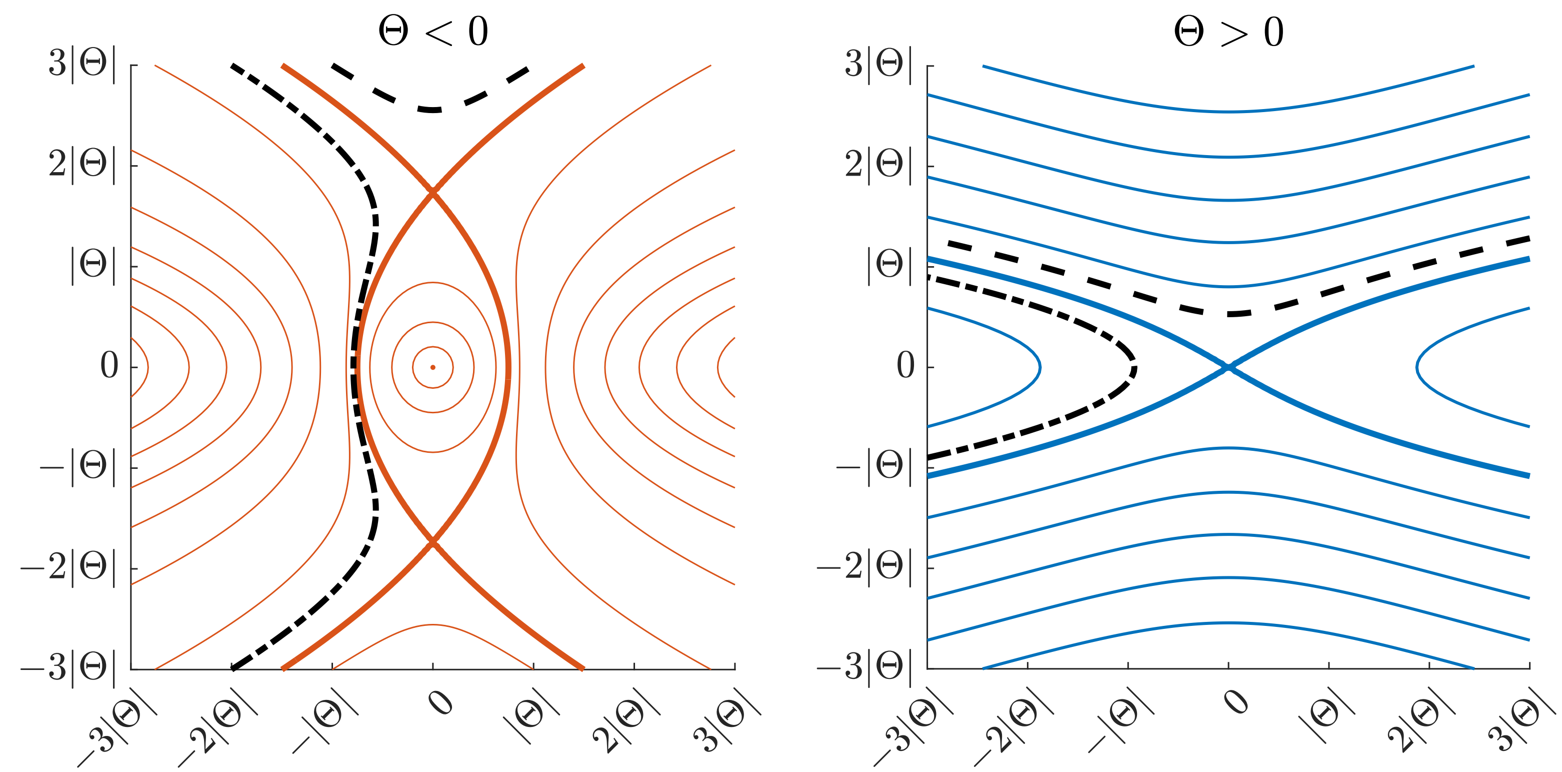


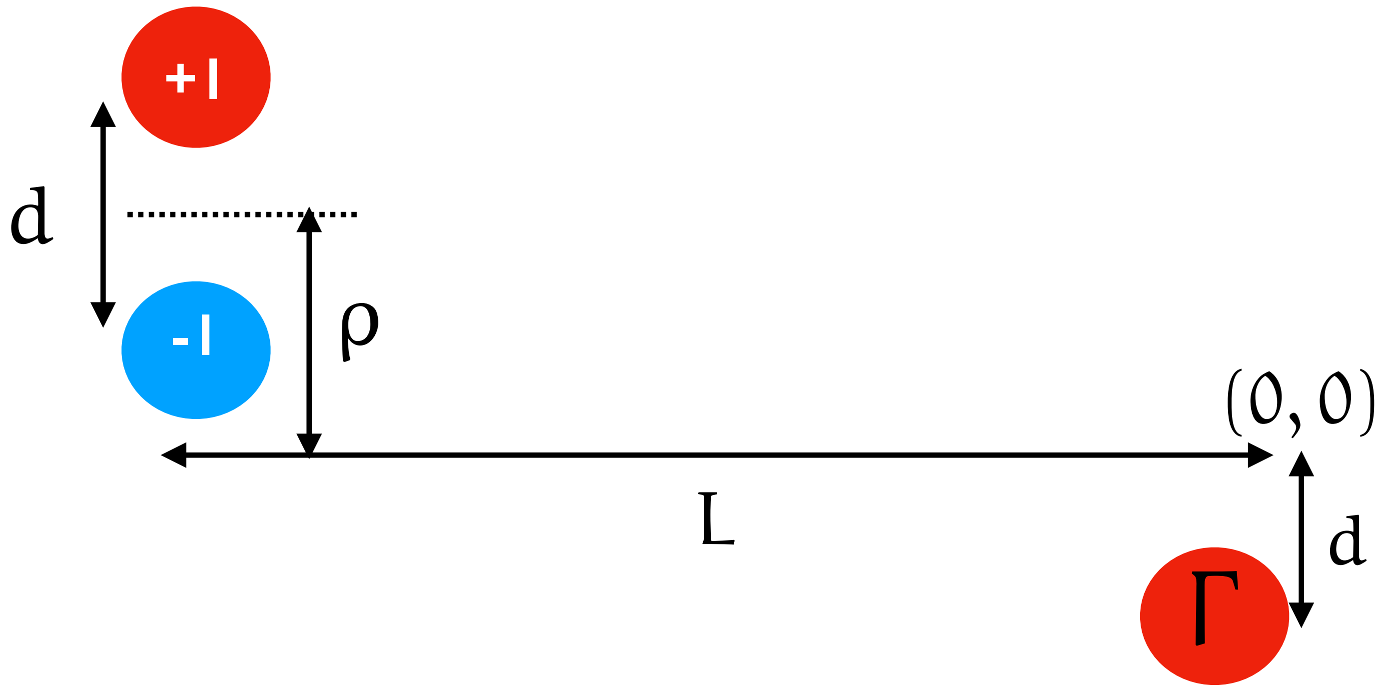
X vanishes when the triangle of vortices is isosceles.

Y vanishes when the three vortices are collinear, which is not a singularity of the coordinate system.

Above the separatrix, $X \to +\infty$ as $t \to +\infty$, $\|v_2\|$ diverges, an *exchange scattering*.

Below the separatrix, $X \to -\infty$ as $t \to +\infty$, $\|v_1\|$ diverges, a *direct scattering*.

Generalizing Dipole Scattering



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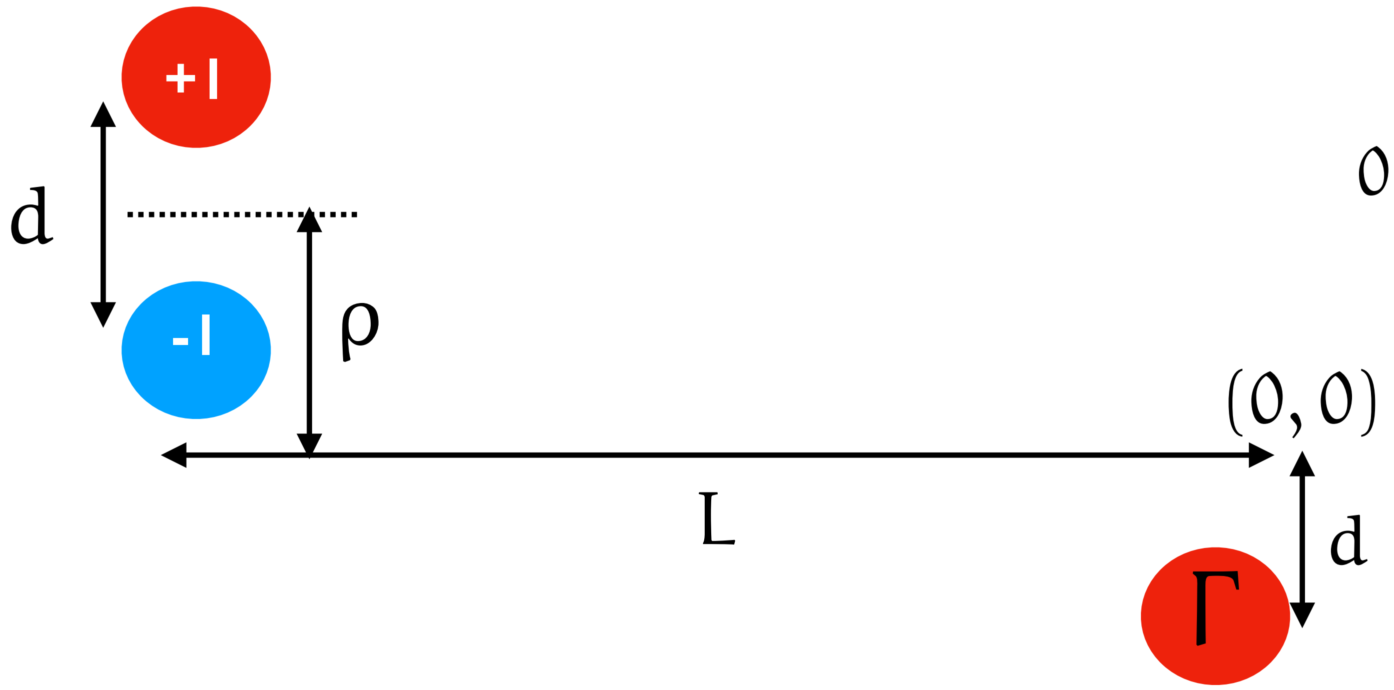

$0 < \Gamma \neq 1$: No Exchange scattering

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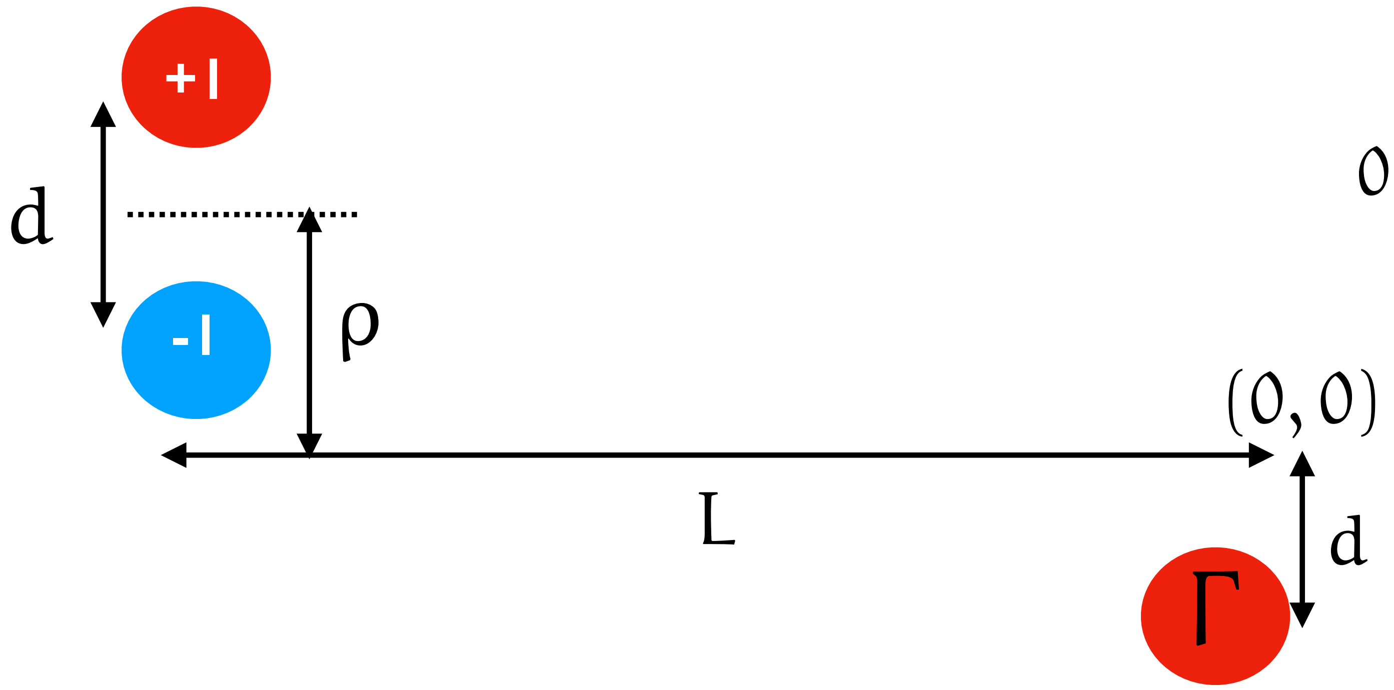

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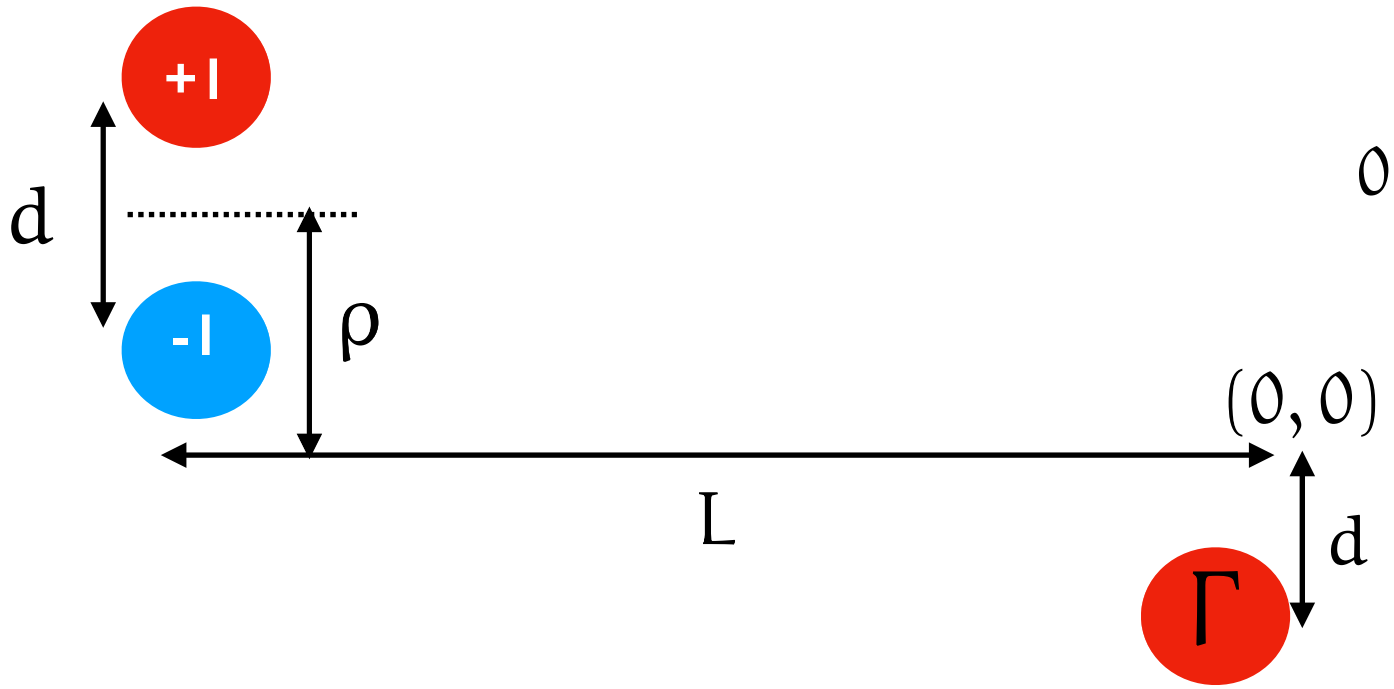
What replaces it?

Generalizing Dipole Scattering



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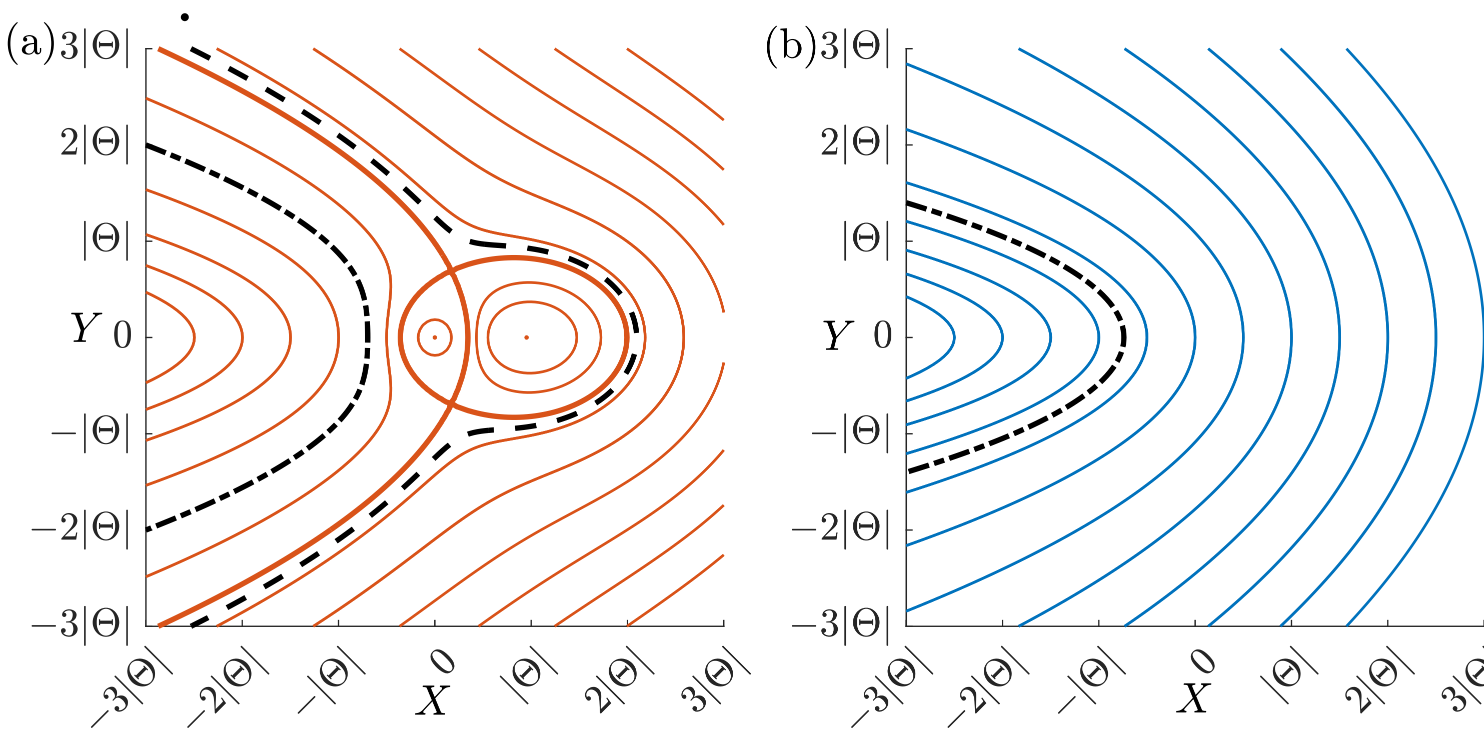
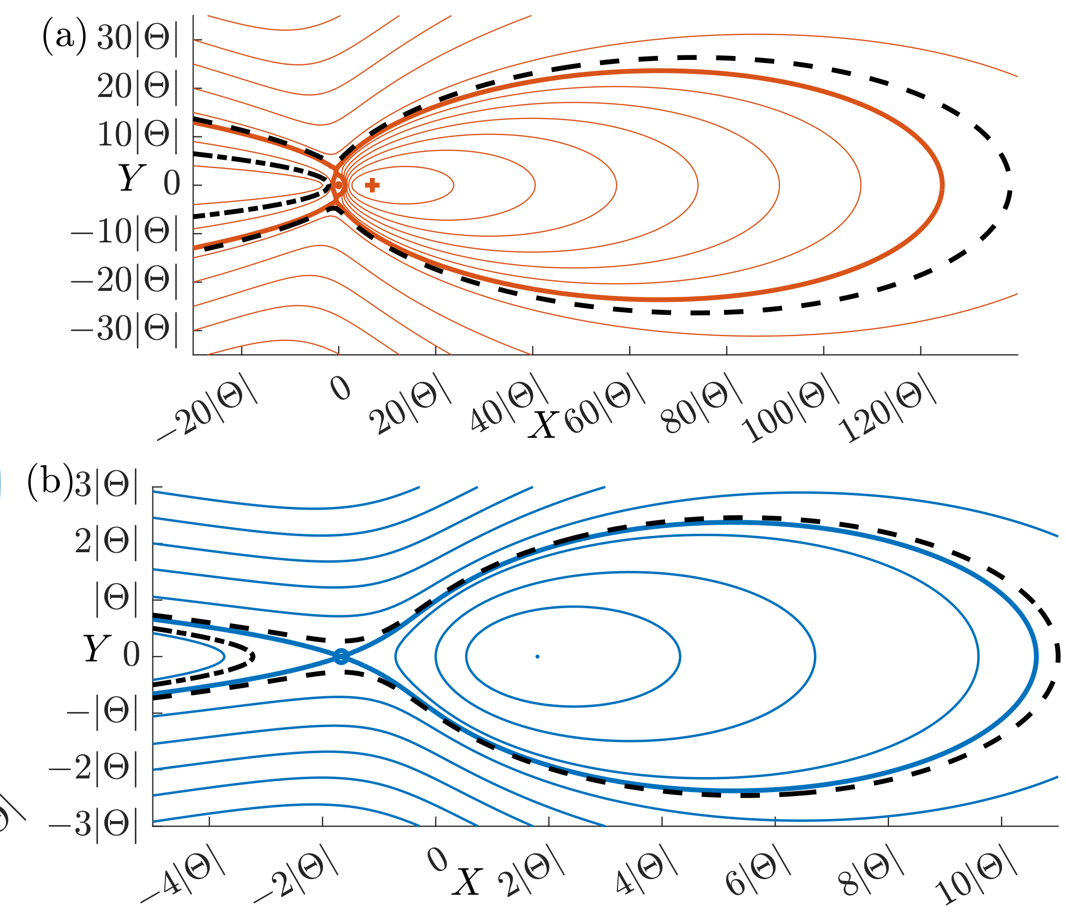
What replaces it?

Extended Direct

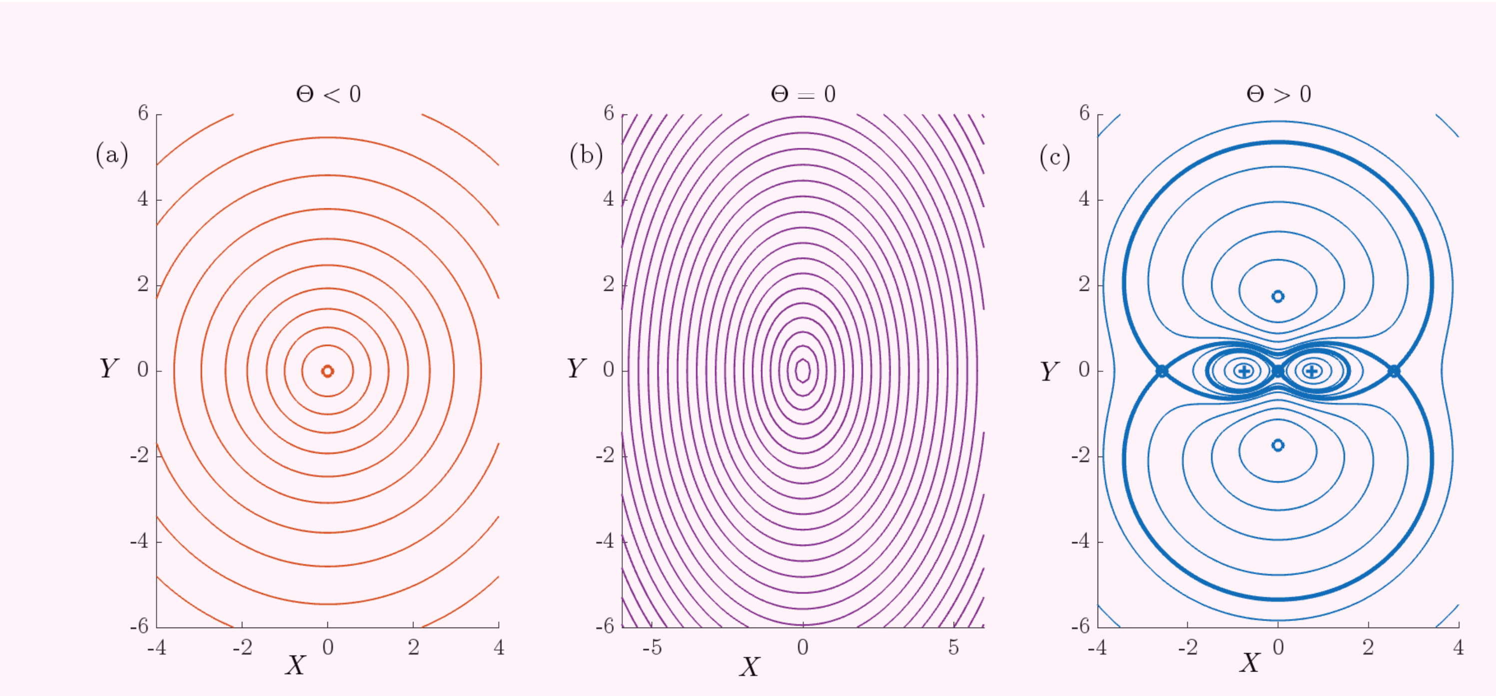


$\Gamma = 0.4$

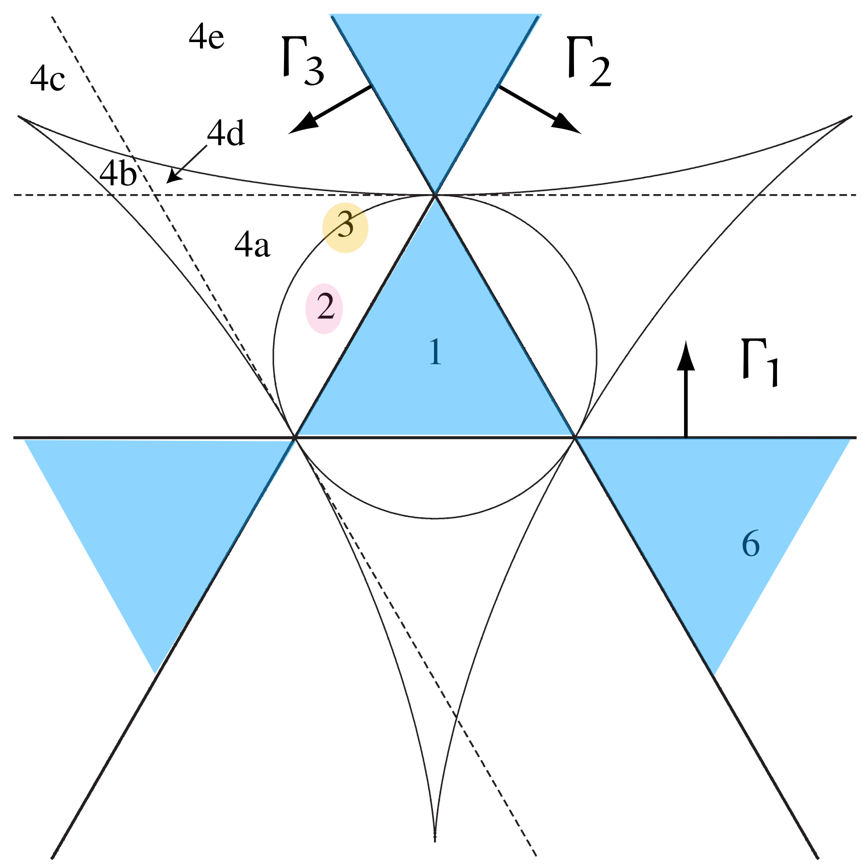


$\Gamma = 1.7$

Global Phase Plane $\kappa_2 < 0$



$$\Gamma = \left(\frac{5}{9}, \frac{5}{9}, -\frac{1}{9}\right)$$



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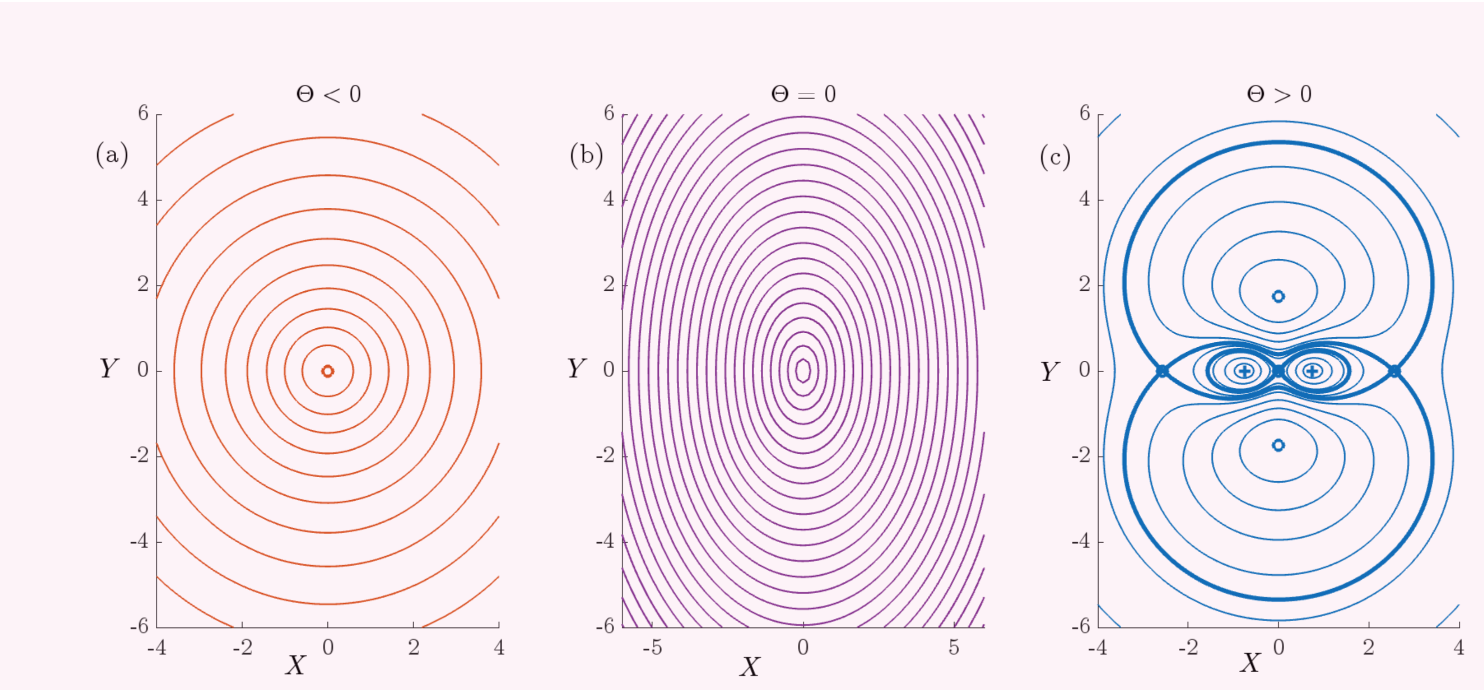

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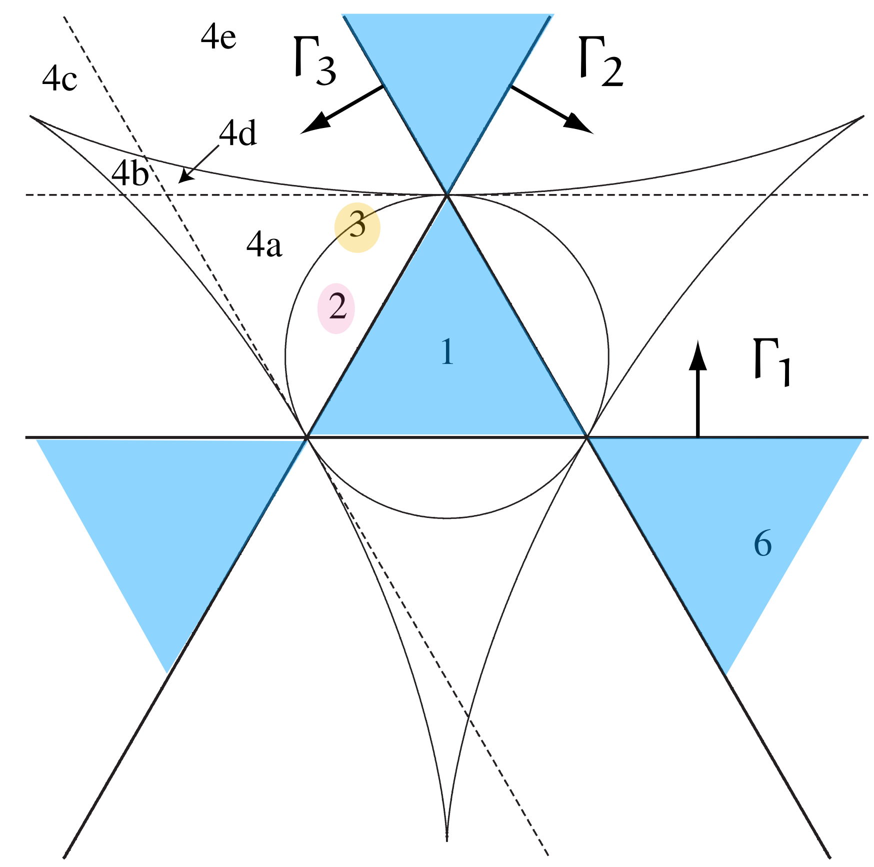
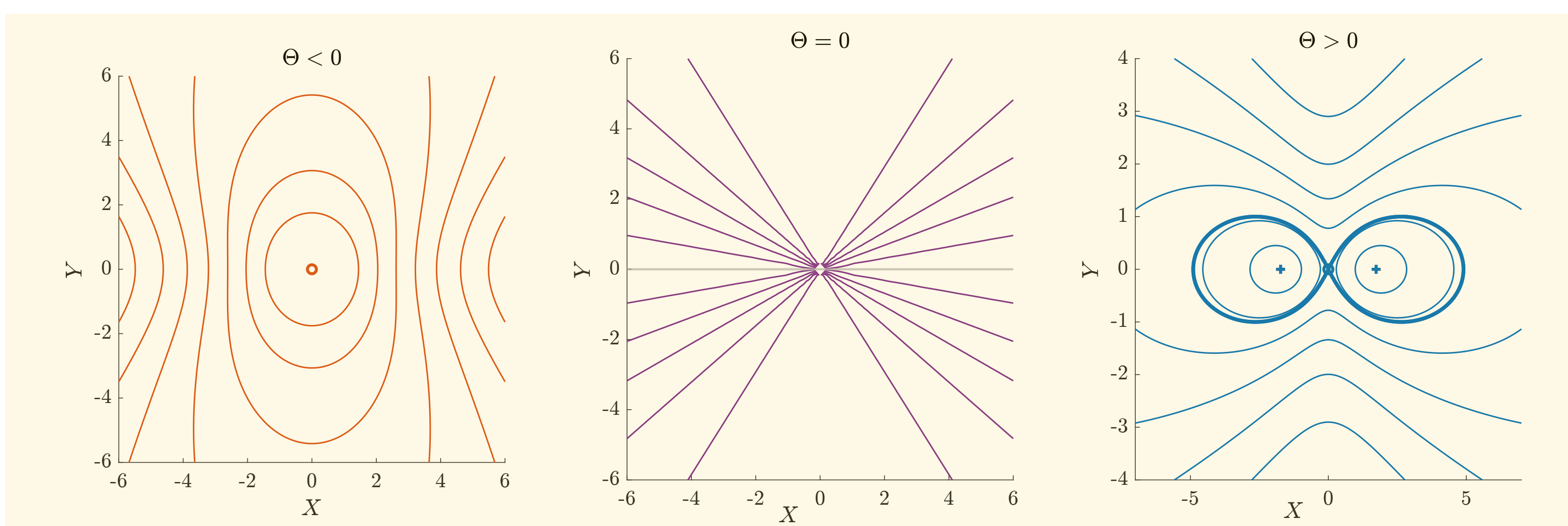


Vortex Collapse

$$\Gamma = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

3 $\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_1\Gamma_3 = 0$



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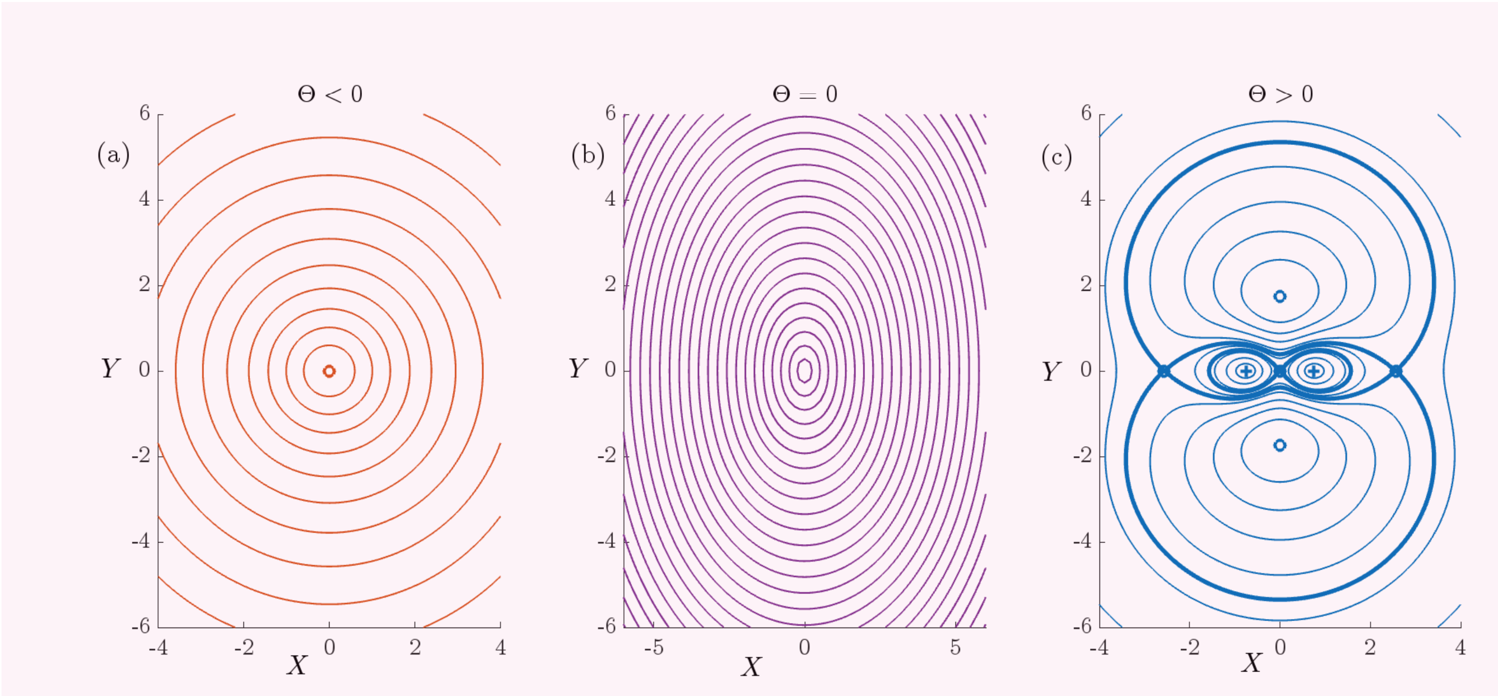

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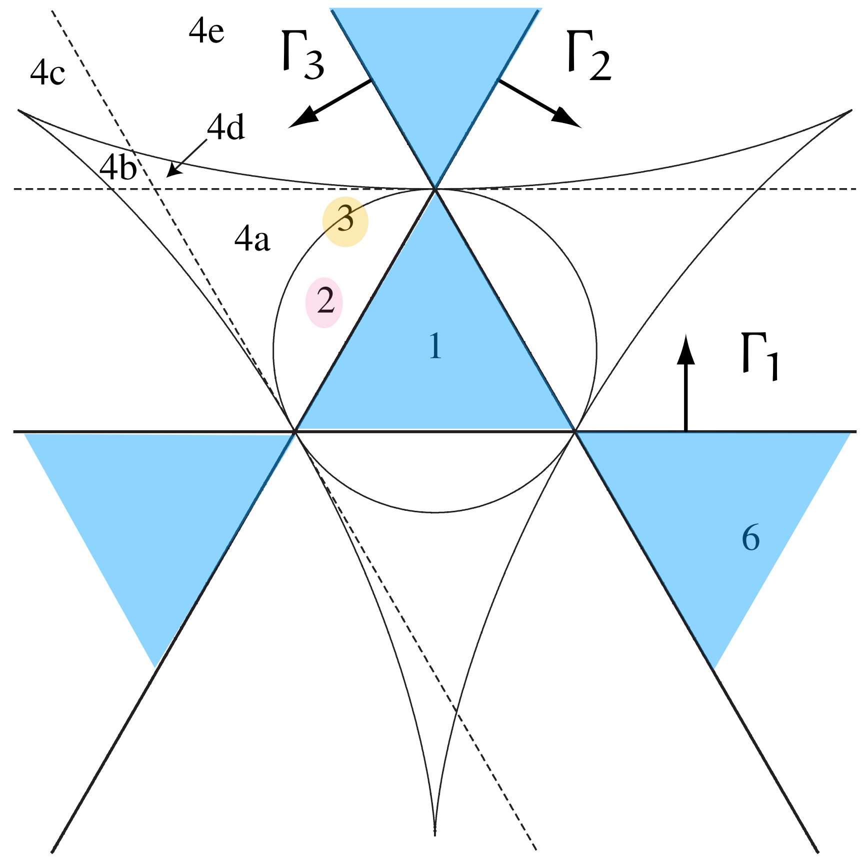
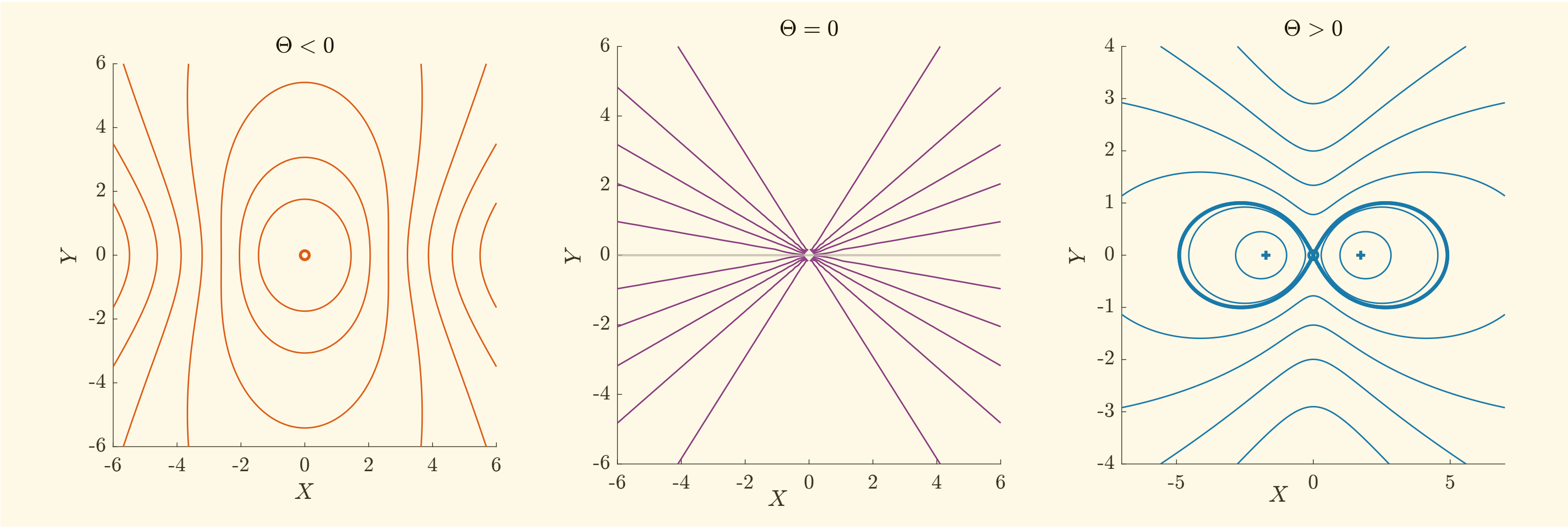


- alters both the topology and the linear stability of all 5 equilibria.
- the linear arrangements become unstable.
- The triangular arrangements in **Region 2** exhibit neutral stability.

Vortex Collapse

$$\Gamma = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

3 $\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_1\Gamma_3 = 0$



Conclusion

- Finish the stability analysis of the three-vortex (arbitrary circulation).
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- Review more literature to find more applications of the new coordinate system.

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Thank you!
Questions?