Math 106: Homework 1

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Due: September 17, 2025

Instructions

Answer all questions carefully. Show your work and include drawings whenever needed. Use complete sentences for descriptive questions. You may hand-draw graphs neatly if preferred.

Part A: Understanding Concepts (2 points each)

Q1. Define the following terms in your own words:

a) Vertex

b) Edge

c) Degree of a vertex

Q2. Explain the difference between a **directed graph** and an **undirected graph**. Provide a small sketch of each.

Q3. State the **Handshaking Lemma** and explain why the number of vertices with odd degree in a graph must always be even.

Q4. What is an **Euler circuit**? Give an example of a small graph that has one.

Q5. What does it mean to **Eulerize** a graph? Why might we want to do this?

Part B: Working with Graphs (3 points each)

Q6. Consider the following graph: Vertices $= \{A, B, C, D\}$, Edges $= \{AB, AC, BD, CD\}$.

a) Draw the graph.

b) List the degree of each vertex.

c) Does this graph have an Euler circuit? Why or why not?

Q7. Given the adjacency matrix below, draw the corresponding graph (undirected):

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Q8. Draw a directed graph with 4 vertices where:

$$\deg^+(A) = 2,\ \deg^-(A) = 1$$

(out-degree of 2, in-degree of 1). Label the edges.

Q9. A graph has vertices A, B, C, D, E with degrees:

$$\deg(A) = 2,\ \deg(B) = 3,\ \deg(C) = 2,\ \deg(D) = 1,\ \deg(E) = 2$$

a) How many edges does the graph have?

b) Is it possible for this graph to have an Euler circuit? Why or why not?

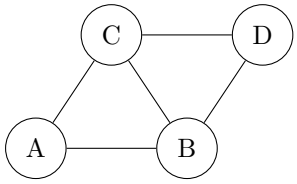
Q10. Draw a graph with 5 vertices that is a **path** but not a **circuit**.

Part C: Euler Circuits and Trails (4 points each)

Q11. For the graph below, list the degree of each vertex. Then determine if the graph has:

a) An Euler circuit

b) An Euler trail (but not a circuit)



Q12. Take the same graph as in Q11. Eulerize the graph by duplicating edges where needed. Show the updated graph.

Q13. Explain why a graph with exactly two vertices of odd degree must have an **Euler trail** but not an Euler circuit.

Q14. Give a real-world example where Euler circuits are useful (for example, garbage collection, mail delivery, or street cleaning). Explain how the problem can be modeled with a graph.

Part D: Adjacency Matrices (4 points each)

Q15. Write the adjacency matrix of the following graph: Vertices: A, B, C, D Edges: AB, AC, BD, CD, DA.

Q16. The adjacency matrix of a directed graph is:

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

a) Draw the graph.

b) Find the in-degree and out-degree of each vertex.

Q17. In your own words, describe one advantage and one disadvantage of using an adjacency matrix compared to an adjacency list.

Part E: Challenge (5 points each)

Q18. For a complete graph with 6 vertices (K_6):

a) How many edges does it have?

b) What is the degree of each vertex?

c) Does K_6 have an Euler circuit? Explain.

Q19. Given an adjacency matrix A, explain what the entry $(A^2)_{ij}$ represents. Then compute $(A^2)_{1,3}$ for:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Section A: Multiple Choice (1 mark each)

1. In an **undirected graph**, the adjacency matrix is always:

   (a) Skew-symmetric

   (b) Symmetric

   (c) Diagonal

   (d) Upper triangular

2. The number of **edges** in a complete graph with n vertices is:

   (a) n^2

   (b) $n(n-1)$

   (c) $\frac{n(n-1)}{2}$

   (d) $n-1$

3. If a vertex has an **out degree of 3** in a directed graph, it means:

   (a) 3 edges go out from it

   (b) 3 edges go into it

   (c) 3 edges are loops

   (d) It's isolated

4. Which graph representation uses **more space** for sparse graphs?

   (a) Adjacency matrix

   (b) Adjacency list

   (c) Edge list

   (d) Tree

5. In an adjacency matrix for an undirected graph, the total number of 1s is:

   (a) Equal to number of edges

   (b) Twice the number of edges

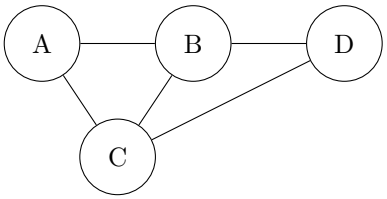
   (c) Half the number of edges

   (d) Always 0

Section B: Short Answer (2–4 marks each)

6. Define the following terms with examples:
   a) Walk
   b) Path
   c) Circuit

7. Given the following graph:



   a) List the set of vertices V and edges E
   b) Draw the **adjacency matrix** (undirected)

8. Given the following **adjacency matrix**, draw the graph (assume undirected):

	A	B	C	D
A	0	1	0	1
B	1	0	1	0
C	0	1	0	1
D	1	0	1	0

9. For the following **directed graph**, write its adjacency matrix:

   Graph: $A \to B,\ A \to C,\ D \to C$

10. Write **two advantages** and **two disadvantages** of using an adjacency matrix to represent graphs.

Section C: Application / Problem Solving (5 marks each)

11. Consider the following adjacency matrix for a **directed graph**:

	A	B	C	D
A	0	1	0	0
B	0	0	1	0
C	0	0	0	1
D	1	0	0	0

   a) Draw the directed graph.
   b) Find the **in-degree** and **out-degree** of each vertex.

12. A graph has the following properties:
   - It has 5 vertices
   - Each vertex is connected to every other vertex

   a) What type of graph is this?
   b) Write the adjacency matrix.
   c) How many edges does it have?