Introduction to Mathematical Modeling

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Semester: Fall 2025 Date: September 4, 2025

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What is a Mathematical Model?

A ***mathematical model*** *is a simplified, structured way to represent a real-world system using mathematical language. It helps us understand, predict, or improve a situation by identifying key relationships, often through equations, graphs, or logical rules.*

Definition (Simplified): A **mathematical model** is a tool that uses math to describe and analyze something in the real world.

Everyday Examples of Models

- **Cost of Apples:** If each apple costs \$2, then buying x apples costs $\boxed{C = 2x}$.
- **Maps:** A map represents geographic locations using symbols and scales to help with navigation.
- **Weather Forecasts:** Meteorologists use models based on atmospheric equations to predict weather.
- **Budgeting:** A personal budget models income and expenses, helping you plan savings and spending.

Why Use Models?

- To test decisions before acting (e.g., "What if I save more each week?")
- To find patterns and make forecasts (e.g., "How will traffic change during rush hour?")
- To optimize outcomes (e.g., "What's the fastest delivery route?")

What is Mathematical Modeling?

Mathematical modeling is the process of building, testing, and using a mathematical model to solve a real-world problem.

The Modeling Process

1. **Understand the Problem:** Define the situation clearly.
2. **Make Assumptions and Simplify:** Focus only on key elements.
3. **Build the Model:** Use math — equations, graphs, tables — to represent it.
4. **Test the Model:** Use real data to see how well it works.
5. **Refine the Model:** Adjust based on accuracy or changes in the real-world system.

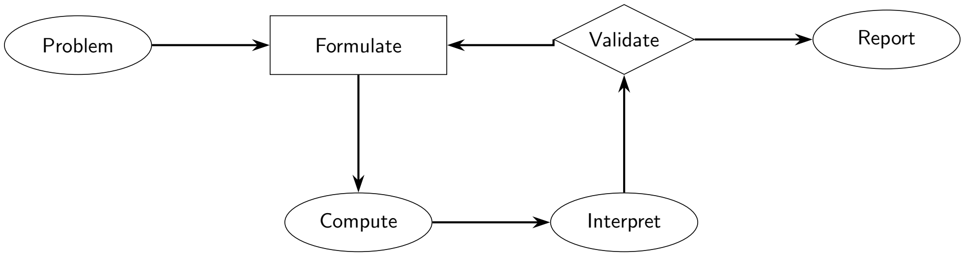

Figure 1: The Mathematical Modeling Process.

Modeling Example: Saving Money Weekly

Question: How much money will I have after saving the same amount every week?

Step 1: Define the Problem

You want to know how much total money you'll save over a certain number of weeks.

Step 2: Make Assumptions & Define Variables

Assumptions:

- The same amount is saved each week.
- No money is withdrawn or lost.

Variables:

- s = money saved each week
- w = number of weeks
- M = total money saved

Step 3: Build the Model

$$M = s \times w$$

Step 4: Apply the Model (Example)

If you save \$5 per week for 4 weeks:

$$M = 5 \times 4 = 20$$

Answer: You will save \$20.

Step 5: Refine if Necessary

If the amount saved varies each week:

$$M = s_1 + s_2 + s_3 + \cdots + s_n$$

Final General Model

$$\boxed{M = s \times w} \quad \text{(if amount per week is constant)}$$

What Will We Model in This Course?

This course will focus on real-life problems you encounter in daily life, business, science, and society. Here are some examples:

1. Delivery and Logistics

- How can companies deliver packages efficiently?
- How can they minimize fuel use or cost?

Concepts:

- Shortest path problems
- Route optimization
- Scheduling deliveries

Delivery and Logistics (Simple Example)

Problem:
A delivery driver starts at a warehouse and needs to drop off packages at three places: A, B, and C. The goal is to find the shortest route to deliver all packages and return to the warehouse.

Step 1: Understand the Problem
We want to help the driver:

- Visit all locations once
- Use the shortest route
- Save fuel and time

Step 2: Make Assumptions
To keep it simple:

- No traffic
- Roads are two-way
- Only distance matters

Step 3: Use a Table of Distances
Here are the distances (in km) between the places:

	W (Start)	A	B	C
W	0	10	15	20
A	10	0	35	25
B	15	35	0	30
C	20	25	30	0

Step 4: Try Different Routes
Let's try a few possible routes and add up the distances:

- W $\to$ A $\to$ C $\to$ B $\to$ W = 10 + 25 + 30 + 15 = **80 km**
- W $\to$ B $\to$ C $\to$ A $\to$ W = 15 + 30 + 25 + 10 = **80 km**
- W $\to$ A $\to$ B $\to$ C $\to$ W = 10 + 35 + 30 + 20 = **95 km**

Step 5: Choose the Best Route
The best (shortest) route is either:

- W $\to$ A $\to$ C $\to$ B $\to$ W
- W $\to$ B $\to$ C $\to$ A $\to$ W

Both give a total of **80 km**, which is the shortest.
Conclusion:
By comparing different delivery paths, we can choose the one that saves the most distance, time, and fuel.

2. Scheduling Tasks

- How do we schedule jobs, meetings, or classes to avoid overlap?
- How do we assign tasks to workers efficiently?

Tools: critical path method, graph theory

Scheduling Tasks (Simple Example)

Problem:
A teacher needs to schedule 3 tasks in a school project. Some tasks must be done before others. The goal is to finish all tasks as soon as possible without overlap.
Step 1: Understand the Problem
We need to:

- Do all tasks in the correct order
- Avoid doing two tasks at the same time (if not allowed)
- Finish as quickly as possible

Step 2: Make Assumptions

- Each task takes a certain number of days
- Some tasks must wait until others are finished

Step 3: List the Tasks and Durations

- Task A: Write report (2 days)
- Task B: Collect data (3 days)
- Task C: Make presentation (1 day)

Task Order:

- Task B must be done **before** Task A
- Task A must be done **before** Task C

Step 4: Make a Schedule
We can now arrange the tasks in order:

- Day 1–3: Task B (Collect data)
- Day 4–5: Task A (Write report)
- Day 6: Task C (Make presentation)

Step 5: Result

- Total time needed: **6 days**
- Tasks were done in the correct order
- No overlap or delay

Conclusion:
By looking at which tasks depend on others, we can build a simple schedule that gets everything done as fast as possible.

3. Maximizing Profit (Optimization)

- A company makes multiple products with limited resources.
- How many of each product should it make to earn the most profit?

Mathematical Ideas:

- Linear programming
- Constraints and feasible regions
- Objective functions

(Optimization)

We can approach this problem using the five-step mathematical modeling process:

1. **Understand the Problem:** The company produces different products, each using a limited amount of resources (e.g., labor, materials, time). Each product yields a specific profit. The goal is to determine how many units of each product to produce in order to maximize total profit, without exceeding the available resources.
2. **Make Assumptions and Simplify:**
   - Only a few key products are considered.
   - Profits and resource usage per product are constant.
   - All products made are sold (no leftovers).
   - Partial products are not allowed (if applicable).

3. **Build the Model:** Use variables to represent the number of each product made.
   - Let x be the number of units of Product A.
   - Let y be the number of units of Product B.
   - Let the profit function be $P = ax + by$, where a and b are the profit per unit of Product A and B respectively.
   - Add constraints for limited resources, such as:

$$c_1x + d_1y \le R_1 \quad \text{(Resource 1 constraint)}$$

$$c_2x + d_2y \le R_2 \quad \text{(Resource 2 constraint)}$$

$$x \ge 0, \quad y \ge 0$$

4. **Test the Model:** Insert real data (values for a, b, c_i, d_i, R_i) to solve the model. Use graphing or algebraic methods (such as linear programming) to find the values of x and y that maximize P.

5. **Refine the Model:** Improve the model based on real-world conditions:
   - Consider more products or additional constraints.
   - Adjust for changes in resource availability or profit margins.
   - Include new conditions (e.g., minimum production, market limits).

This process helps convert a real-world profit question into a mathematical optimization problem that can be analyzed and solved.

Simple Example: Maximizing Profit

Problem:
A company makes two products: **Chairs** and **Tables**.

- Each Chair gives a profit of \$30.
- Each Table gives a profit of \$50.
- Each Chair requires 2 units of wood and 3 hours of labor.
- Each Table requires 4 units of wood and 2 hours of labor.
- Available resources: 100 units of wood, 90 hours of labor.

Step 1: Define Variables
Let:

$$x = \text{number of Chairs}, \quad y = \text{number of Tables}$$

Step 2: Objective Function (Profit)

$$\text{Maximize } P = 30x + 50y$$

Step 3: Constraints

$$\begin{aligned} \text{Wood constraint:} \quad & 2x + 4y \le 100 \\ \text{Labor constraint:} \quad & 3x + 2y \le 90 \\ \text{Non-negativity:} \quad & x \ge 0, \quad y \ge 0 \end{aligned}$$

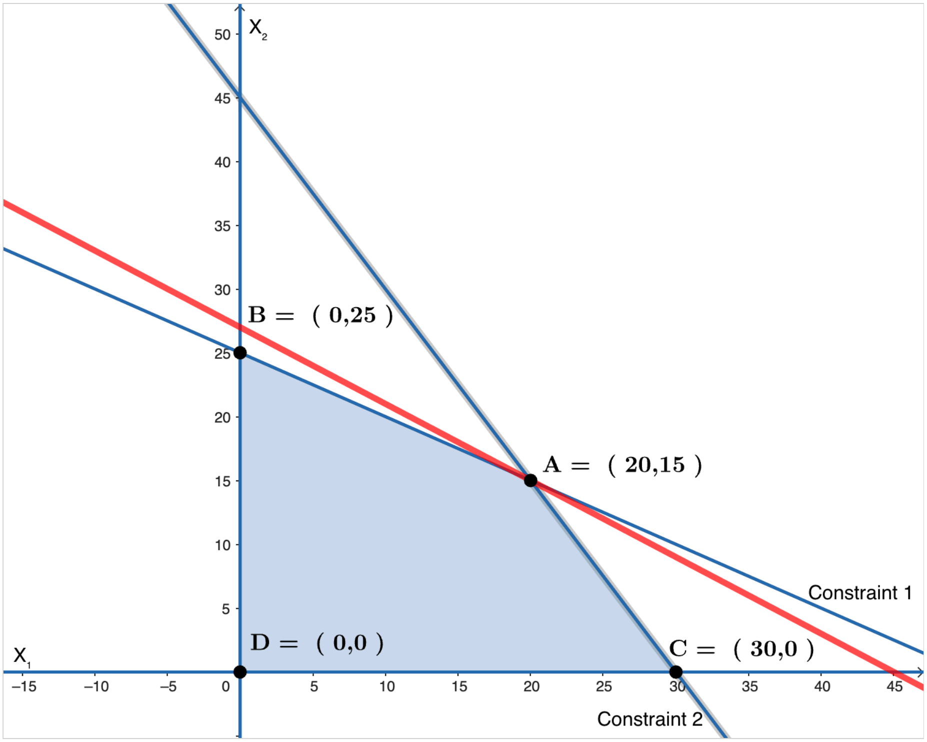

Figure 2: Graphical Solution where shaded region shows the feasible area

Step 4: Solve Graphically or Algebraically
Points A, B, C, D are the possible solutions for x and y., we want to pick the point that maximizes the profit.

Step 5: Optimal Solution (from solving):

$$x = 20, \quad y = 15$$

$$\text{Maximum Profit: } P = 30(20) + 50(15) = \boxed{1350}$$

4. Traffic Control

- **How can we reduce traffic congestion?**
- **How should traffic lights be timed to improve traffic flow?**

Why does this matter? Traffic congestion leads to wasted time, increased fuel consumption, environmental harm, and driver frustration. Effective traffic control can improve public safety, reduce pollution, and save money.

Modeling Traffic with Mathematics

Mathematical modeling helps city planners and engineers design better traffic systems by simulating how vehicles move and interact. Below are the key concepts and tools used in traffic modeling.

Figure 3: Traffic Congestion

1. Traffic Flow Model

Traffic flow can be modeled using the fundamental relationship:

$$\text{Flow} = \text{Density} \times \text{Speed}$$

- **Flow:** Number of vehicles passing a point per unit time (e.g., cars per minute)
- **Density:** Number of vehicles per unit length of road (e.g., cars per mile)
- **Speed:** Average speed of the vehicles (e.g., miles per hour)

As density increases, speed typically decreases, which can lead to a reduction in flow if the road becomes too crowded — this is how traffic jams form.

5. Fair Division and Apportionment

- How do we fairly divide resources, like seats in Congress?
- What if items can't be divided equally?

Concepts:

- Apportionment methods (Hamilton, Jefferson)
- Fair division algorithms

6. Game Strategy and Decision Making

- What is the best strategy in a competitive game?
- Can certain games be "solved" with logic?

7. Tracking Goods and Systems

- How do barcodes and scanning systems track packages?
- How can we reduce errors in inventory tracking?

Why is Mathematical Modeling Important?

- It helps you make better decisions with evidence.
- It clarifies complex systems through simplification.
- It improves efficiency, fairness, and performance.

Whether you're in business, public policy, engineering, or education — modeling empowers you to tackle real problems logically.

You Don't Need to Be a Math Expert

This course is designed for everyone. No advanced math is required.

What you'll need:

- An open and curious mindset
- Willingness to try new problem-solving methods
- Comfort with experimenting and learning from mistakes

The goal isn't just to solve math problems — it's to solve real-life problems using math.

Modeling is where math meets meaning. Welcome aboard!

End of Lecture #2