Introduction to Mathematical Modeling

Ramapo College of New Jersey

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Semester: Fall 2025 Date: September 8, 2025

Graph Theory

Euler Circuits: Walking Every Path Once

Objective

- Understand what an Euler Circuit is
- Learn when a graph has an Euler Circuit or Euler Path
- Explore real-life applications of Euler Circuits
- Practice analyzing graphs for Euler properties

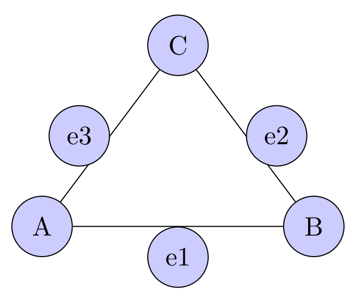
Introduction and History

Leonhard Euler, considered one of the greatest mathematicians, first studied these circuits in 1736 while solving the famous *Seven Bridges of Königsberg* problem. He asked if it was possible to walk through the city crossing each bridge exactly once and return to the starting point.

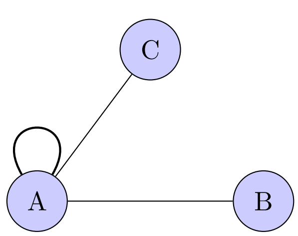
Question: Why is this problem interesting? What real-world issues might require such analysis?

Basic Definitions

- **Graph:** A graph is a picture made of dots (called *vertices*) and lines (called *edges*) that connect some of the dots.

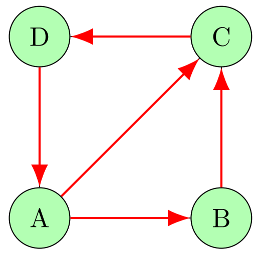


- **Degree of a Vertex:** The number of edges incident to the vertex. (Loops count twice since they start and end at the same vertex.)



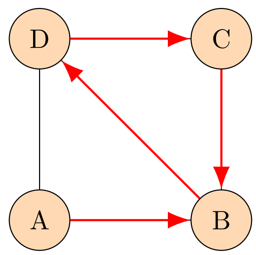
Degree of vertex A: 4 (3 edges incident, loop counts twice)

- **Euler Path:** A path that traverses every edge exactly once (may start and end at different vertices).

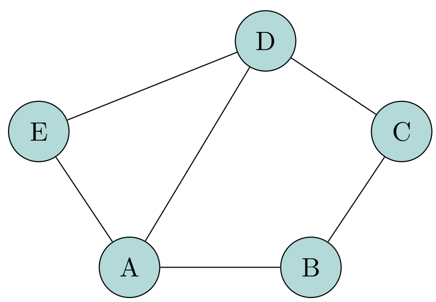


Euler Circuit: A path that starts and ends at the same vertex, visiting every edge exactly once (shown in red with arrows).

- **Euler Circuit (Eulerian Circuit):** An Euler Path that starts and ends at the same vertex.



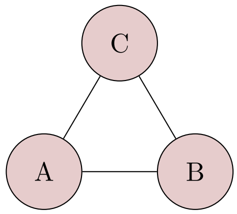
- **Connected Graph:** A graph where every vertex is reachable from every other vertex by some path.



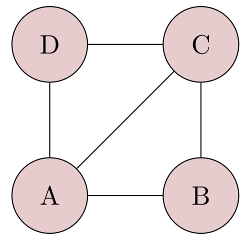
- **Disconnected Graph:** A disconnected graph is a graph in which at least one pair of vertices is not reachable from one another by any path.

Disconnected Graph:
A graph is disconnected if it is not connected, meaning the vertices are split into two or more groups (called components) with no edges connecting these groups.

Examples and Intuition



Triangle Graph: Is there an Euler Circuit? Why?



Square with Diagonal: Does it have an Euler Path or Euler Circuit?

Eulers Theorem (Eulers Criterion)

Theorem: A connected graph has an Euler Circuit if and only if **every vertex has even degree**.
A connected graph has an Euler Path (but no Euler Circuit) if and only if **exactly two vertices have odd degree**.

If there are more than two vertices with odd degree, then no Euler Path or Circuit exists.

Discussion: Why do you think the degree of vertices matters for Euler Paths? Think about entering and leaving a vertex edges must "pair up."

Sketch of the Proof

Necessity (Only If):

- When traveling along edges, each time you enter a vertex you must also leave it (except possibly start/end vertices).
- So, vertices on an Euler Circuit must have an even number of edges (even degree).
- For an Euler Path (start and end differ), exactly two vertices have odd degree.

Sufficiency (If):

- You can build the Euler Circuit by:
- Starting anywhere
- Following unused edges until you return to start
- If some edges remain, find a vertex in the path with unused edges and repeat (Hierholzers algorithm)

Applications of Euler Circuits

- **Route Planning:** Mail carriers delivering mail without retracing streets.
- **DNA Sequencing:** Reconstructing sequences from fragment overlaps.
- **Electrical Circuits:** Designing paths for wiring without duplication.
- **Puzzle Solving:** Games like the Königsberg bridge problem and the Seven Bridges puzzles.

Summary

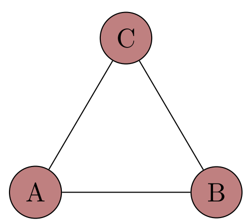
- Euler Circuits allow traversing every edge exactly once.
- Degree of vertices is the key criterion.
- Real-world problems often reduce to checking Eulerian properties.

Problem Set: Euler Circuits

Instructions: For each graph below:

1. List the degree of each vertex.
2. Determine whether an Euler Circuit exists.
3. If so, describe a possible Euler Circuit (order of vertices).

Problem 1: Triangle Graph

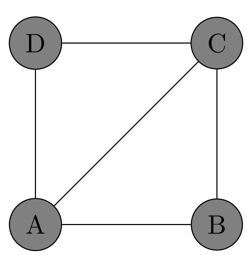


Your Answer:

- Degrees:
- Euler Circuit? ____________
- Path: ____

Degrees: A=2, B=2, C=2
Euler Circuit? Yes (all even)

Problem 2: Square with Diagonal

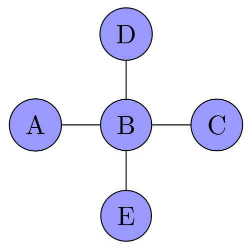


Your Answer:

- Degrees:
- Euler Circuit? ______________
- Path: ____

Degrees: A=3, B=2, C=3, D=2
Euler Circuit? No (vertices A and C have odd degree)
No Euler Circuit exists

Problem 3: Cross Shape (+)

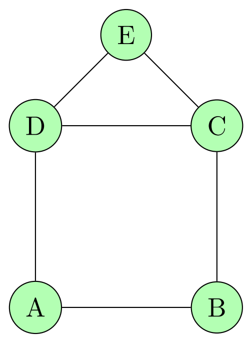


Your Answer:

- Degrees:
- Euler Circuit? ______________
- Path: ____

Degrees: A=1, B=4, C=1, D=1, E=1
Euler Circuit? No (A, C, D, E have odd degree)
No Euler Circuit exists

Problem 4: House Graph

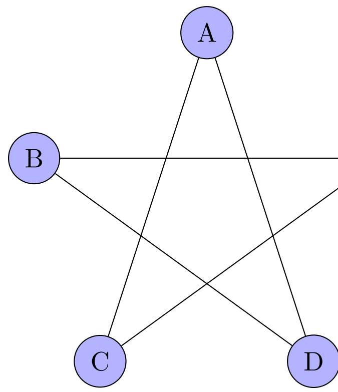


Your Answer:

- Degrees:
- Euler Circuit? ______________
- Path: ____

Degrees: A=2, B=2, C=3, D=3, E=2
Euler Circuit? No (C and D have odd degree)
No Euler Circuit exists

Problem 5: Star Shape

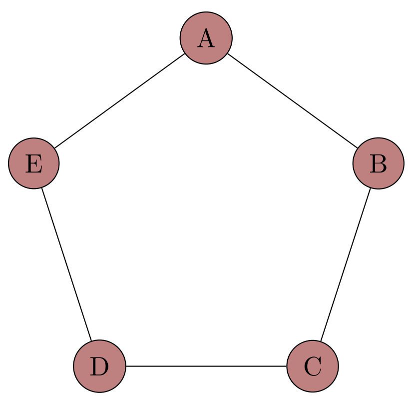


Your Answer:

- Degrees:
- Euler Circuit? ____________
- Path: ____

Degrees: A=2, B=2, C=2, D=2, E=2
Euler Circuit? Yes (all even)

Problem 6: Pentagon Graph

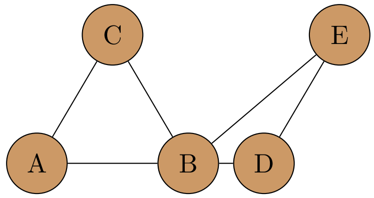


Your Answer:

- Degrees:
- Euler Circuit? ____________
- Path: ____

Degrees: A=2, B=2, C=2, D=2, E=2
Euler Circuit? Yes (all even)

Problem 7: Bowtie Graph



Your Answer:

- Degrees:
- Euler Circuit? ________________
- Path: ____

Degrees: A=2, B=4, C=2, D=2, E=2
Euler Circuit? Yes (all even)

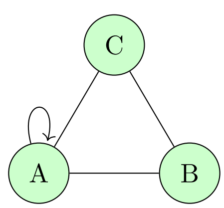
Problem 8: Disconnected Graph

Your Answer:

- Degrees:
- Euler Circuit? ________________
- Path: ____

Degrees: A=1, B=1, C=1, D=2, E=1
Euler Circuit? No (graph disconnected and odd degrees)

Problem 9: Graph with a Loop

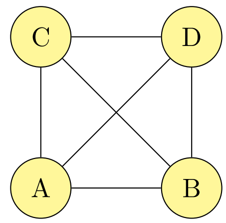


Your Answer:

- Degrees:
- Euler Circuit? ________________
- Path: ____

Degrees: A=4 (loop counts as 2), B=2, C=2
Euler Circuit? Yes (all even)

Problem 10: Complete Graph K4

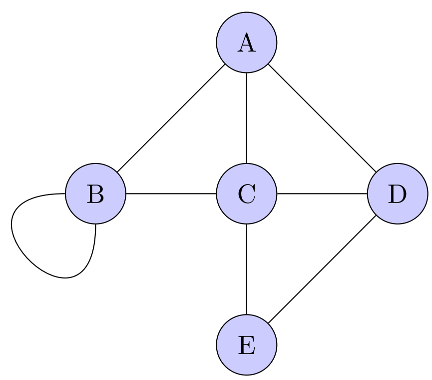


Your Answer:

- Degrees:
- Euler Circuit? ____________
- Path: ______________________________________

Problem

Consider the following graph:



Answer the following questions:

(a) How many vertices are in this graph?

(b) How many edges are in this graph? (Count the loop at B as one edge.)

(c) State the degree of the vertices in the graph in non-increasing order.

(d) Does an Euler circuit exist in this graph? Justify your answer.

Solutions

(a) The number of vertices in the graph is 5 (vertices A, B, C, D, E).

(b) The number of edges in the graph is 8.

- Edges: $A - B, A - C, A - D, B - C, C - D, C - E, D - E$ (7 edges)
- Plus the loop at vertex B counts as 1 edge.

Total edges $= 7 + 1 = 8$.

(c) Degrees of vertices (remember loops count twice for degree):

- $\deg(A) = 3$ (connected to B, C, D)
- $\deg(B) = 4$ (edges: $A - B, B - C$, plus loop counts as 2)
- $\deg(C) = 4$ (edges: $A - C, B - C, C - D, C - E$)
- $\deg(D) = 3$ (edges: $A - D, C - D, D - E$)
- $\deg(E) = 2$ (edges: $C - E, D - E$)

Sorted in non-increasing order:

$$4, 4, 3, 3, 2$$

(d) **Does an Euler circuit exist?**

An Euler circuit exists if and only if the graph is connected and every vertex has an even degree.

- The graph is connected (every vertex can be reached from any other vertex). - Vertex A has degree 3 (odd), vertex D has degree 3 (odd), and vertex E has degree 2 (even).

Since not all vertices have even degree, **an Euler circuit does NOT exist**.

End of Lecture #2