Introduction to Mathematical Modeling

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Introduction to Graph Theory

A **graph** is a mathematical structure used to model pairwise relations between objects. It is defined as:

$$G = (V, E)$$

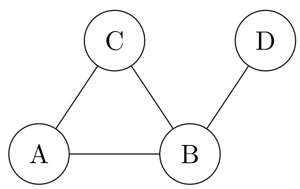
where:

- V is a non-empty set of **vertices** (or **nodes**)
- E is a set of **edges**, each connecting a pair of vertices

Undirected Graph

In an **undirected graph**, edges have no direction. That is, an edge $\{u, v\} \in E$ indicates a bi-directional relationship between vertices u and v. The edge set E is a set of **unordered** pairs of vertices.

Example:



Here:

$$V = \{A, B, C, D\}, \quad E = \{\{A, B\}, \{A, C\}, \{B, C\}, \{B, D\}\}$$

Degrees:

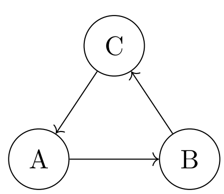
$$\deg(A) = 2, \quad \deg(B) = 3, \quad \deg(C) = 2, \quad \deg(D) = 1$$

Total degree sum: $2 + 3 + 2 + 1 = 8 \Rightarrow \frac{8}{2} = 4$ edges

Directed Graph

In a **directed graph** (or **digraph**), edges have a specific direction. An edge $(u, v) \in E$ indicates a directed connection from vertex u to vertex v. Here, the edge set E is a set of **ordered** pairs.

Example:



Here:

$$V = \{A, B, C\}, \quad E = \{(A, B), (B, C), (C, A)\}$$

Degrees:

$$\deg^+(A) = 1, \quad \deg^-(A) = 1, \deg^+(B) = 1, \quad \deg^-(B) = 1, \deg^+(C) = 1, \quad \deg^-(C) = 1$$

Vertex Degree

In an undirected graph, the **degree** of a vertex v, denoted $\deg(v)$, is the number of edges incident to v. If a loop exists at v, it contributes **2** to $\deg(v)$, since it connects to v twice.
In a directed graph:

- The **in-degree** $\deg^{-}(v)$ is the number of edges coming *into* v
- The **out-degree** $\deg^{+}(v)$ is the number of edges going *out of* v

The Handshaking Lemma

Statement: In any undirected graph, the sum of the degrees of all vertices is equal to twice the number of edges.

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof Sketch: Each edge contributes exactly 1 to the degree of both of its endpoints. Therefore, every edge is counted twice in the sum of degrees.
Corollary: In every undirected graph, the number of vertices with an *odd* degree is even.

Example (Handshaking Lemma):

Given a graph with vertex degrees:

$$\deg(A) = 1, \quad \deg(B) = 2, \quad \deg(C) = 3, \quad \deg(D) = 2, \quad \deg(E) = 2$$

Total degree sum:

$$1 + 2 + 3 + 2 + 2 = 10 \Rightarrow \frac{10}{2} = 5 \text{ edges}$$

Number of vertices with odd degree: A (1) and C (3): 2 vertices, Even

Definition: Euler Circuit

An **Euler circuit** is a closed trail (a path that starts and ends at the same vertex) in a graph that visits every edge exactly once.
A graph has an Euler circuit **if and only if**:

- The graph is **connected**, and
- All vertices have **even degree**.

Definition: Eulerizing a Graph

Eulerizing a graph means modifying it so that it contains an Euler circuit. This is done by:

- Adding edges **between existing vertices** (allowing duplicates),
- Until all vertices have **even degree**.

Note: Edges are added only between vertices that already have a path no new vertices are introduced.

Key Rule:

The number of vertices with odd degree in any graph is always **even**. Therefore, Eulerizing involves pairing up odd-degree vertices and connecting them with duplicate edges.

Step-by-Step Process to Eulerize a Graph

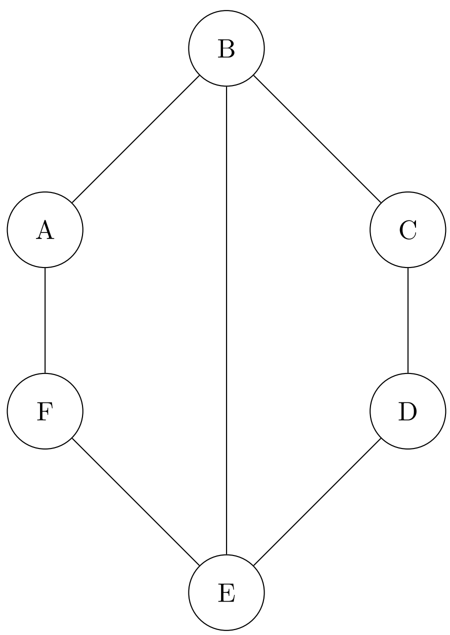
1. Identify all **odd-degree vertices**.
2. Find the **shortest path** between any pair of odd-degree vertices.
3. Add that path to the graph (duplicate the edges along the path).
4. Repeat steps 1-3 until all vertices have even degrees.

Example Graph

Vertices: A, B, C, D, E, F
Edges:

A–B, B–C, C–D, D–E, E–F, F–A, B–E

Initial Graph



Step 1: Identify Odd-Degree Vertices

- deg(A) $= 2$
- deg(B) $= 3 \rightarrow$ **odd**
- deg(C) $= 2$

- deg(D) = 2
- deg(E) = 3 → **odd**
- deg(F) = 2

Odd-degree vertices: B, E

Step 2: Find Shortest Path Between Odd-Degree Vertices

The shortest path between B and E is the direct edge **B–E** (already exists).

Step 3: Add the Path to the Graph

Duplicate the edge B–E:

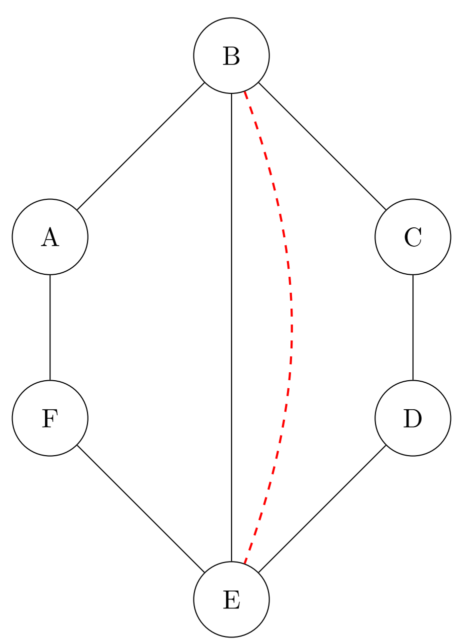
- deg(B): 3 → 4
- deg(E): 3 → 4

Step 4: Confirm All Degrees Are Even

- All vertices now have even degree.
- Graph remains connected.

Conclusion: The graph is now **Eulerized** and contains an Euler circuit.

Updated Graph with Duplicated Edge (Dashed in Red)



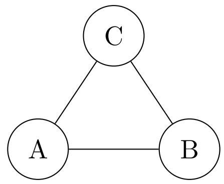
5. Adjacency Matrix

The **adjacency matrix** of a graph with n vertices is an $n \times n$ matrix $A = [a_{ij}]$, where:

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge from vertex } i \text{ to vertex } j \\ 0, & \text{otherwise} \end{cases}$$

5.1 Undirected Graph Example

Consider this undirected graph with vertices A, B, and C:



Label vertices: $A = 1$, $B = 2$, $C = 3$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

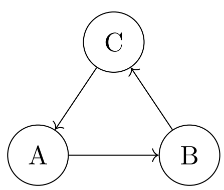
Note:

- The matrix is **symmetric** since the graph is undirected.
- Diagonal entries are 0 (no loops).
- Row sums (or column sums) give vertex degrees.

$$\deg(A) = 2, \quad \deg(B) = 2, \quad \deg(C) = 2.$$

5.2 Directed Graph Example

Now consider a directed graph:



Label: $A = 1$, $B = 2$, $C = 3$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Note:

- The matrix is **not symmetric**.
- Row i shows edges **from** vertex i (outgoing)

- Column j shows edges **to** vertex j (incoming)

$$\text{Out-degrees: } \deg^+(A) = 1,\ \deg^+(B) = 1,\ \deg^+(C) = 1$$
$$\text{In-degrees: } \deg^-(A) = 1,\ \deg^-(B) = 1,\ \deg^-(C) = 1$$

5.3 Properties of Adjacency Matrices

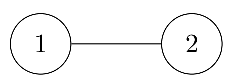
- For an undirected graph:
  - A is symmetric
  - The sum of all elements in A is $2|E|$
- For a directed graph:
  - The sum of row $i = \deg^+(v_i)$
  - The sum of column $j = \deg^-(v_j)$
- For both types:
  - Diagonal entries a_{ii} count loops
  - You can compute the number of paths of length k using powers of A: $(A^k)_{ij}$ = number of walks from i to j of length k

1. Can a Matrix Entry Be 2?

Yes! This happens if two points are joined by more than one edge.

Example: Simple Graph

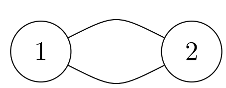
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Here, the 1 means there is one edge between vertices 1 and 2.

Example: Multigraph

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$



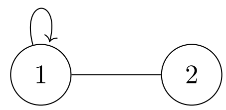
Here, the 2 means there are two edges between vertices 1 and 2.

What If There Is a Loop?

A loop is when a vertex connects to itself. In the matrix, loops appear on the diagonal (the entries A_{ii}).

Example: Graph with a Loop

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



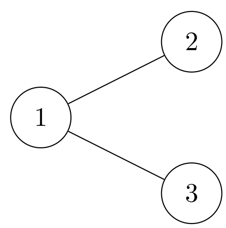
- The 1 in the top-left means vertex 1 has a loop.
- The 1 in position $(1, 2)$ means there is an edge from 1 to 2.

Can the Matrix Be Used to Count Connections?

Yes! The number of edges connected to a vertex is called its degree.

Undirected Graph Example

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



- Row 1 has two 1s $\Rightarrow$ vertex 1 has degree 2.
- Row 2 has one 1 $\Rightarrow$ vertex 2 has degree 1.
- Row 3 has one 1 $\Rightarrow$ vertex 3 has degree 1.

Worksheet: For each graph, write the adjacency matrix.

Figure A

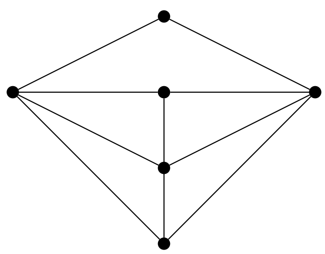


Figure B

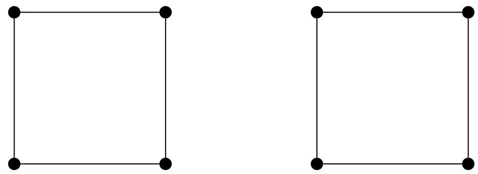


Figure C

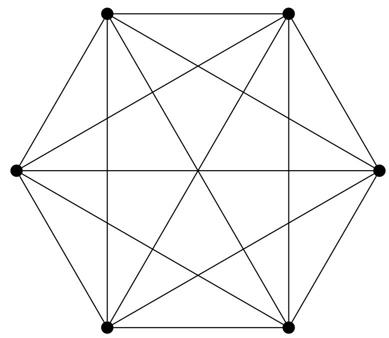


Figure D

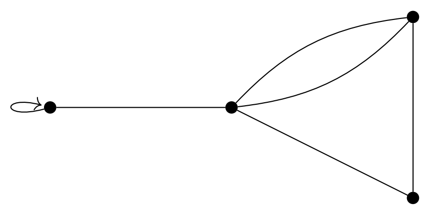
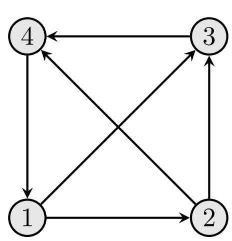


Figure E



For each adjacency matrix, draw the graph.

Matrix A

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Matrix B

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix C

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Matrix D

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Matrix E

$$E = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

End of Lecture #3