Introduction to Mathematical Modeling

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Introduction

We begin our discussion with an example of a furniture dealer, which will lead us to a mathematical formulation of a Linear Programming Problem (LPP) involving two variables.

Example: Furniture Dealer Problem

A furniture dealer can invest in buying tables, chairs, or a combination of both. His goal is to maximise his profit while considering the following constraints:

- He has a maximum capital of Rs 50,000.
- He can store at most 60 items (tables + chairs).

Investment and Profit Details

- Cost of one table = Rs 2500; Profit per table = Rs 250
- Cost of one chair = Rs 500; Profit per chair = Rs 75

Some Investment Scenarios

- Buying only tables:

$$\text{Maximum number of tables} = \frac{50000}{2500} = 20$$
$$\text{Profit} = 250 \times 20 = \text{Rs } 5000$$

- Buying only chairs:

$$\text{Maximum number of chairs (by capital)} = \frac{50000}{500} = 100$$
$$\text{But can store only 60 chairs}$$
$$\text{Profit} = 75 \times 60 = \text{Rs } 4500$$

- Buying 10 tables and 50 chairs:

$$\text{Total items} = 10 + 50 = 60 \quad \text{(Storage constraint satisfied)}$$
$$\text{Total cost} = 10 \times 2500 + 50 \times 500 = 25000 + 25000 = 50000 \quad \text{(Investment constraint satisfied)}$$
$$\text{Profit} = 10 \times 250 + 50 \times 75 = 2500 + 3750 = \text{Rs } 6250$$

Clearly, different strategies yield different profits. The key question is:
How should the dealer invest to get the maximum profit?

Mathematical Formulation of the Problem

Let:

$$x = \text{number of tables purchased}$$
$$y = \text{number of chairs purchased}$$

Constraints

- Non-negativity constraints:
$$x \geq 0 \quad \text{and} \quad y \geq 0 \tag{1, 2}$$
- Investment constraint:
$$2500x + 500y \leq 50000 \Rightarrow 5x + y \leq 100 \tag{3}$$
- Storage constraint:
$$x + y \leq 60 \tag{4}$$

Objective Function

The dealer wants to maximise profit, Z, given by:

$$Z = 250x + 75y \tag{5}$$

Complete Mathematical Model

Maximise:

$$Z = 250x + 75y$$

Subject to:

$$5x + y \leq 100$$
$$x + y \leq 60$$
$$x \geq 0, \quad y \geq 0$$

This is a **Linear Programming Problem** (LPP).

Definition of Terms

- **Linear Programming Problem (LPP)**: A problem that seeks to optimise (maximise or minimise) a linear objective function subject to a set of linear inequalities or constraints.
- **Objective Function**: A linear function of decision variables (e.g., $Z = ax + by$) that needs to be maximised or minimised. In our example, $Z = 250x + 75y$ is the objective function.
- **Decision Variables**: Variables whose values are to be determined. Here, x and y are decision variables.
- **Constraints**: Restrictions or conditions (usually in the form of linear inequalities) that the decision variables must satisfy. In this example, constraints include investment and storage limitations.
- **Non-Negativity Constraints**: Conditions that require all decision variables to be greater than or equal to zero, i.e., $x \geq 0, y \geq 0$.
- **Optimisation Problem**: A problem that involves finding the best solution (maximum or minimum) from a set of feasible solutions.

Problem 1: Maximise $Z = 3x + 4y$

Subject to:

$$\begin{aligned} x + y &\leq 4 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Step 1: Convert inequality to equality to find boundary

Line: $x + y = 4$

- When $x = 0 \Rightarrow y = 4$
- When $y = 0 \Rightarrow x = 4$

This line intersects the axes at $(0, 4)$ and $(4, 0)$.

Step 2: Identify the feasible region

Also considering:

$$x \geq 0 \quad \text{(x-axis)} \quad \& \quad y \geq 0 \quad \text{(y-axis)}$$

The feasible region is the triangle bounded by:

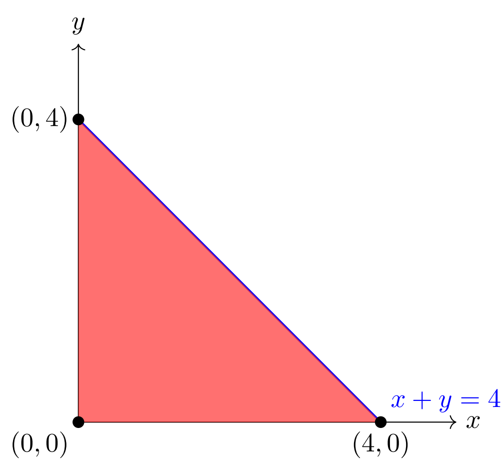
$$(0,0), \quad (0,4), \quad (4,0)$$

Step 3: Evaluate the objective function $Z = 3x + 4y$

$$\begin{aligned} Z(0,0) &= 3(0) + 4(0) = 0 \\ Z(0,4) &= 3(0) + 4(4) = 16 \\ Z(4,0) &= 3(4) + 4(0) = 12 \end{aligned}$$

Maximum value of $Z = 16$ **occurs at** $(0, 4)$

Step 4: Graphical Solution



Problem 2: Minimise $Z = -3x + 4y$

Subject to:

$$\begin{aligned} x + 2y &\leq 8 \\ 3x + 2y &\leq 12 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Step 1: Convert inequalities to equalities to find boundaries

Line 1: $x + 2y = 8$

- When $x = 0 \Rightarrow y = 4$
- When $y = 0 \Rightarrow x = 8$

Line 2: $3x + 2y = 12$

- When $x = 0 \Rightarrow y = 6$
- When $y = 0 \Rightarrow x = 4$

Step 2: Find intersection of the two lines

$$\begin{aligned} x + 2y &= 8 \quad (1) \\ 3x + 2y &= 12 \quad (2) \end{aligned}$$

Subtract (1) from (2):

$$(3x + 2y) - (x + 2y) = 12 - 8 \Rightarrow 2x = 4 \Rightarrow x = 2$$

Substitute $x = 2$ into equation (1):

$$2 + 2y = 8 \Rightarrow 2y = 6 \Rightarrow y = 3$$

Intersection point: $(2, 3)$

Step 3: Identify the corner points of the feasible region

From the axes and constraint boundaries, we identify the feasible region is bounded by:

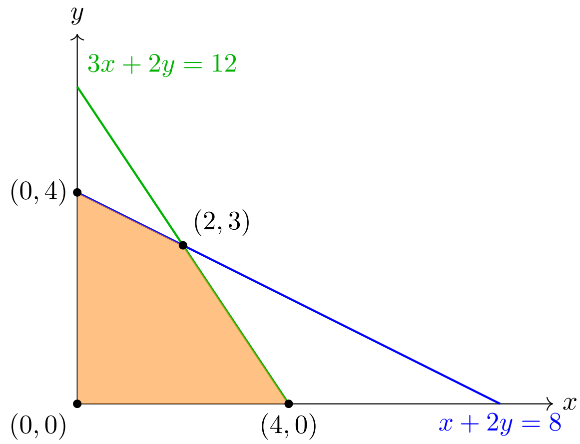
$$(0,0), \quad (0,4), \quad (2,3), \quad (4,0)$$

Step 4: Evaluate the objective function $Z = -3x + 4y$

$$\begin{aligned} Z(0,0) &= -3(0) + 4(0) = 0 \\ Z(0,4) &= -3(0) + 4(4) = 16 \\ Z(2,3) &= -3(2) + 4(3) = -6 + 12 = 6 \\ Z(4,0) &= -3(4) + 4(0) = -12 \end{aligned}$$

Minimum value of $Z = -12$ **occurs at** $(4, 0)$

Step 5: Graphical Solution



Problem: Minimise $Z = x + 2y$

Subject to:

$$\begin{aligned} 2x + y &\geq 3 \\ x + 2y &\geq 6 \\ x, y &\geq 0 \end{aligned}$$

Step 1: Boundary Lines

1. $2x + y = 3$

- $x = 0 \Rightarrow y = 3$
- $y = 0 \Rightarrow x = 1.5$

2. $x + 2y = 6$

- $x = 0 \Rightarrow y = 3$
- $y = 0 \Rightarrow x = 6$

Step 2: Intersection Point of Constraints

Solve:

$$\begin{aligned} 2x + y &= 3 \quad (1) \\ x + 2y &= 6 \quad (2) \end{aligned}$$

From (1): $y = 3 - 2x$
Substitute into (2):

$$x + 2(3 - 2x) = 6 \Rightarrow x + 6 - 4x = 6 \Rightarrow -3x = 0 \Rightarrow x = 0, \quad y = 3$$

So, intersection point is $(0, 3)$

Step 3: Feasible Region and Corner Points

Feasible region is above both lines and in the first quadrant. The key corner points are:

$$(0,3), \quad (6,0)$$

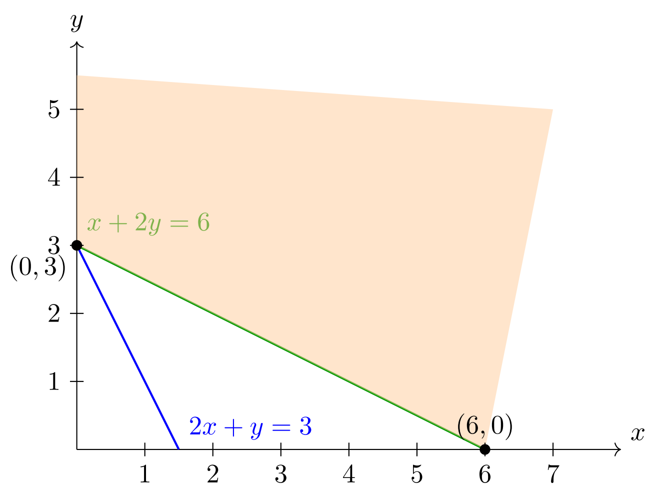
Region is unbounded, but these are the lowest feasible points.

Step 4: Evaluate $Z = x + 2y$

$$Z(0,3) = 0 + 2(3) = 6$$
$$Z(6,0) = 6 + 0 = 6$$

Minimum value of $Z = 6$ **occurs at both** $(0,3)$ **and** $(6,0)$

Step 5: Graphical Solution

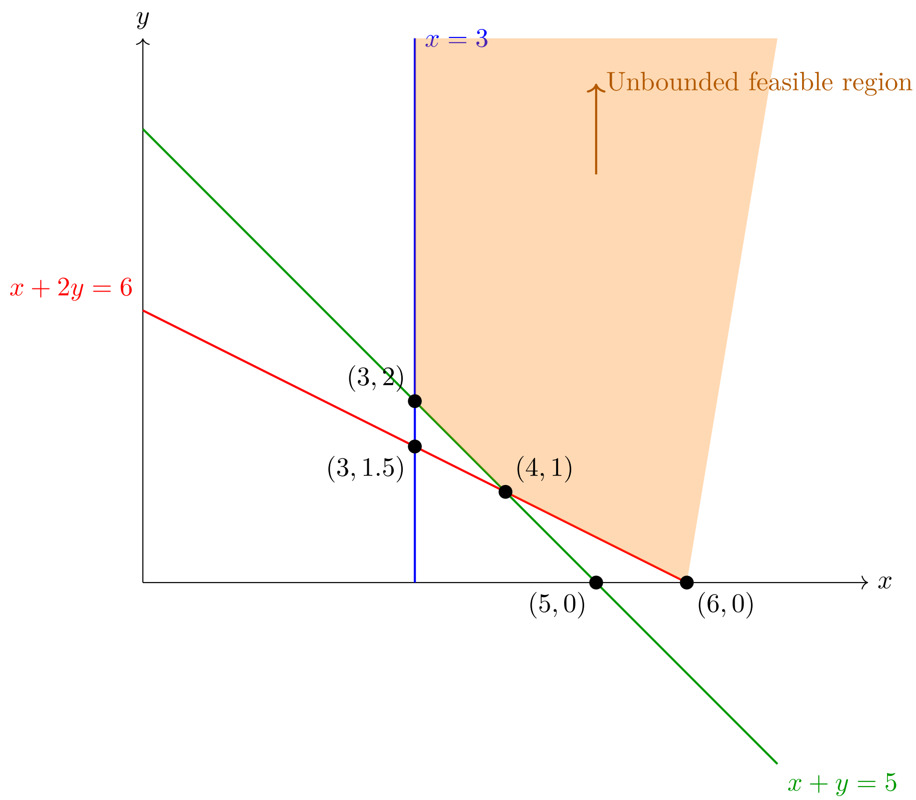


Problem 3: Maximize $Z = -x + 2y$

Subject to:

$$\begin{cases} x \geq 3 \\ x + y \geq 5 \\ x + 2y \geq 6 \\ y \geq 0 \end{cases}$$

Observation: The feasible region is unbounded upward and extends infinitely in y, so $Z = -x + 2y$ can increase without bound by increasing y.



Conclusion: The feasible region extends infinitely upward, and since $Z = -x + 2y$ increases as $y \to \infty$, the problem has **no finite maximum**.

Problem 4

A firm manufactures two types of products A and B, and sells them at a profit of Rs. 2 on product A and Rs. 3 on product B. Each product is processed on two machines M_1 and M_2. Product A requires 1 minute on M_1 and 2 minutes on M_2. Product B requires 1 minute on M_1 and 1 minute on M_2. Machine M_1 is available for at most 6 hours 40 minutes, and machine M_2 is available for 10 hours in any working day. Formulate this as a linear programming problem (LPP).

Solution. Let

$$x = \text{number of units of product } A, \qquad y = \text{number of units of product } B.$$

The time constraints:
- On M_1: $1 \cdot x + 1 \cdot y \leq 6 \text{ h } 40 \text{ min} = 400$ minutes.
- On M_2: $2 \cdot x + 1 \cdot y \leq 10 \text{ h} = 600$ minutes.

Nonnegativity:

$$x \geq 0, \quad y \geq 0.$$

Profit (objective function):

$$Z = 2x + 3y \quad \text{(to be maximized)}.$$

Hence the LPP is:

$$\begin{aligned} \text{Maximize } Z &= 2x + 3y, \\ \text{subject to } x + y &\le 400, \\ 2x + y &\le 600, \\ x \ge 0,\ y &\ge 0. \end{aligned}$$

Problem 5

A firm manufactures two products, each of which must be processed through two departments, 1 and 2. The hourly requirements per unit for each product in each department, the weekly capacities in each department, the selling price per unit, labour cost per unit, and rawmaterial cost per unit are given as follows:

	Product A	Product B	Weekly capacity
Dept. 1 (hours per unit)	3	2	130
Dept. 2 (hours per unit)	4	6	260
Selling price ($)	25	30	
Labour cost ($)	16	20	
Raw material cost ($)	4	4	

Formulate a linear programming problem to determine how many units of each product to make in order to maximize total contribution.

Solution outline.
Let

$$x = \text{number of units of product A}, \qquad y = \text{number of units of product B}.$$

Compute the contribution (profit) per unit:

$$\text{Contribution on } A = 25 - 16 - 4 = 5, \qquad \text{Contribution on } B = 30 - 20 - 4 = 6.$$

So the objective function is:

$$\max Z = 5x + 6y.$$

Now the capacity constraints:
- Department 1:

$$3x + 2y \ \le \ 130.$$

- Department 2:

$$4x + 6y \ \le \ 260.$$

Also nonnegativity:

$$x \ge 0,\ y \ge 0.$$

Thus the LPP is:

$$\begin{aligned} \text{Maximize } Z &= 5x + 6y, \\ \text{subject to } 3x + 2y &\le 130, \\ 4x + 6y &\le 260, \\ x \ge 0,\ y &\ge 0. \end{aligned}$$

End of Lecture #7