Introduction to Mathematical Modeling

Ramapo College of New Jersey

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Linear Programming Word Problems

Problem

A firm manufactures two types of products A and B, and sells them at a profit of \$2 on product A and \$3 on product B. Each product is processed on two machines M_1 and M_2. Product A requires 1 minute on M_1 and 2 minutes on M_2. Product B requires 1 minute on M_1 and 1 minute on M_2. Machine M_1 is available for at most 6 hours 40 minutes, and machine M_2 is available for 10 hours in any working day. Formulate this as a linear programming problem (LPP).

Solution. Let

$$x = \text{number of units of product } A, \qquad y = \text{number of units of product } B.$$

The time constraints:
- On M_1: $1 \cdot x + 1 \cdot y \leq 6$ h 40 min $= 400$ minutes.
- On M_2: $2 \cdot x + 1 \cdot y \leq 10$ h $= 600$ minutes.

Nonnegativity:

$$x \geq 0, \quad y \geq 0.$$

Profit (objective function):

$$Z = 2x + 3y \quad \text{(to be maximized).}$$

Hence the LPP is:

$$\begin{aligned} \text{Maximize } & Z = 2x + 3y, \\ \text{subject to } & x + y \leq 400, \\ & 2x + y \leq 600, \\ & x \geq 0,\ y \geq 0. \end{aligned}$$

Problem

A firm manufactures two products, each of which must be processed through two departments, 1 and 2. The hourly requirements per unit for each product in each department, the weekly capacities in each department, the selling price per unit, labour cost per unit, and rawmaterial cost per unit are given as follows:

	Product A	Product B	Weekly capacity
Dept. 1 (hours per unit)	3	2	130
Dept. 2 (hours per unit)	4	6	260
Selling price (\$)	25	30	
Labour cost (\$)	16	20	
Raw material cost (\$)	4	4	

Formulate a linear programming problem to determine how many units of each product to make in order to maximize total contribution.

Solution outline.

Let

$$x = \text{number of units of product A}, \qquad y = \text{number of units of product B}.$$

Compute the contribution (profit) per unit:

$$\text{Contribution on } A = 25 - 16 - 4 = 5, \qquad \text{Contribution on } B = 30 - 20 - 4 = 6.$$

So the objective function is:

$$\max Z = 5x + 6y.$$

Now the capacity constraints:
- Department 1:

$$3x + 2y \ \leq \ 130.$$

- Department 2:

$$4x + 6y \ \leq \ 260.$$

Also nonnegativity:

$$x \geq 0, \ y \geq 0.$$

Thus the LPP is:

$$\begin{aligned} \text{Maximize } Z &= 5x + 6y, \\ \text{subject to } 3x + 2y &\leq 130, \\ 4x + 6y &\leq 260, \\ x \geq 0, \ y &\geq 0. \end{aligned}$$

Problem

A manufacturer makes two types of toys A and B. Three machines are needed for this purpose, and the time (in minutes) required for each toy on each machine is given below:

Type of Toy	Machine I	Machine II	Machine III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is 7.50 and that on each toy of type B is 5, show that 15 toys of type A and 30 toys of type B should be manufactured in a day to get maximum profit.

Solution

Let x = number of type$A$ toys produced, and y = number of type$B$ toys produced.

Constraints from machine times

- **Machine I constraint:** Each A takes 12 minutes, each B takes 6 minutes. Machine I is available for $6 \times 60 = 360$ minutes.

$$12x + 6y \ \leq 360 \quad \Longrightarrow \quad 2x + y \ \leq 60.$$

- **Machine II constraint:** Each A takes 18 minutes, B takes 0 minutes.

$$18x + 0 \cdot y \ \leq 360 \quad \Longrightarrow \quad x \ \leq 20.$$

- **Machine III constraint:** Each A takes 6 minutes, each B takes 9 minutes.

$$6x + 9y \le 360 \implies 2x + 3y \le 120.$$

Also $x \ge 0,\ y \ge 0$.

Objective function

Profit $Z = 7.50x + 5y$. We want to max Z subject to the above constraints.

Cornerpoint method

The feasible regions corner (extreme) points are found to be:

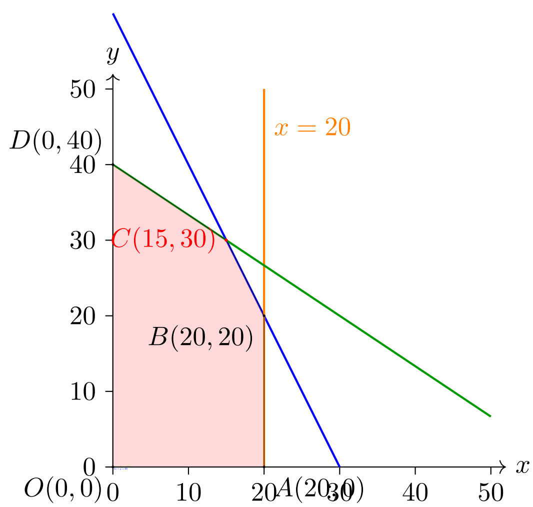
$$O(0,0),\ A(20,0),\ B(20,20),\ C(15,30),\ D(0,40).$$

Compute $Z = 7.50x + 5y$ at each:

Point	(x,y)	$Z = 7.50x + 5y$
O	$(0,0)$	0
A	$(20,0)$	$7.50 \cdot 20 + 5 \cdot 0 = 150$
B	$(20,20)$	$7.50 \cdot 20 + 5 \cdot 20 = 250$
C	$(15,30)$	$7.50 \cdot 15 + 5 \cdot 30 = 112.5 + 150 = 262.5$
D	$(0,40)$	$7.50 \cdot 0 + 5 \cdot 40 = 200$

The maximum value of Z is $\boxed{262.5}$ (i.e. 262.50), achieved at $(x,y) = (15,30)$.
Thus, 15 toys of type A and 30 toys of type B should be manufactured in a day for maximum profit.

Graphical Solution



The shaded region is the feasible region bounded by the constraints:

$$\begin{aligned} 2x + y &\le 60, \\ x &\le 20, \\ 2x + 3y &\le 120, \\ x, y &\ge 0. \end{aligned}$$

The maximum profit of 262.50 is achieved at point $(15,30)$.

Problem

Maximize

$$Z = px + qy, \quad p, q > 0,$$

subject to the constraints:

$$\begin{cases} x + y \leq 60, \\ 5x + y \leq 100, \\ x \geq 0, \\ y \geq 0. \end{cases}$$

(i) Solve graphically to find the corner points of the feasible region.

(ii) It is given that $Z = px + qy$ is maximum at $(0, 60)$ and $(10, 50)$. Find the relation between p and q. Also state how many optimal solutions are possible in that case.

Solution

(i) Corner points of the feasible region

The constraints in the positive quadrant define a feasible polygon whose boundary lines are:

1. $x + y = 60$, that is $y = 60 - x$. 2. $5x + y = 100$, that is $y = 100 - 5x$. 3. The axes $x = 0$ and $y = 0$.

Intersecting these gives corner (extreme) points:

- Intersection of $x + y = 60$ with $y = 0$: $(60, 0)$. - Intersection of $5x + y = 100$ with $y = 0$: $5x = 100 \implies x = 20$. So $(20, 0)$. - Intersection of $x + y = 60$ and $5x + y = 100$:

$$\begin{cases} x + y = 60, \\ 5x + y = 100. \end{cases}$$

Subtract first from second: $5x - x = 100 - 60 \implies 4x = 40 \implies x = 10$. Then $y = 60 - 10 = 50$. So $(10, 50)$. - Intersection of $x + y = 60$ with $x = 0$: $(0, 60)$. - Intersection of $5x + y = 100$ with $x = 0$: $(0, 100)$, but that lies outside $x + y \leq 60$, so not in the feasible region. - Also the origin $(0, 0)$ is a corner.

Thus the feasible regions corners are

$$O = (0, 0), \ A = (0, 60), \ B = (10, 50), \ C = (20, 0).$$

(ii) Condition for multiple maxima

We are given that the objective $Z = px + qy$ attains its maximum value both at $(0, 60)$ and $(10, 50)$.

That means:

$$p \cdot 0 + q \cdot 60 \ = \ p \cdot 10 + q \cdot 50.$$

That is,

$$60q = 10p + 50q \quad \implies \quad 60q - 50q = 10p \quad \implies \quad 10q = 10p \quad \implies \quad p = q.$$

So the necessary relation is $p = q$.

When $p = q$, the objective line $Z = px + qy = p(x + y)$ is proportional to $x + y$. In that case, all points on the line segment joining $(0, 60)$ and $(10, 50)$ inside the feasible region will give the same maximum Z. Hence there are infinitely many optimal solutions (i.e. the entire line segment between those two corner points is optimal).

End of Lecture #8