Introduction to Mathematical Modeling

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UPC Codes and Check Digits

UPC codes use a **check digit** to minimize scanning errors. A *check digit* is an extra digit included in a code to help detect errors.

For a UPC code

$$a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12},$$

the last digit a_{12} (the check digit) is chosen so that the sum

$$3(a_1 + a_3 + a_5 + a_7 + a_9 + a_{11}) + (a_2 + a_4 + a_6 + a_8 + a_{10})$$

is divisible by 10.

Example: What is the check digit for the UPC code `0 6 4 1 4 4 2 8 2 6 3 __` (the last digit unknown)?

Modular Arithmetic and Congruences

The numbers 20, 60, and 100 are all divisible by 10. This leads us to a way of talking about numbers when we only care about their remainders.

Definition 1 (Congruence Modulo m)**.** *Let a, b, and m be integers with $m \geq 2$. Then*

$$a \equiv b \pmod{m}$$

means that m divides $a - b$.

Determine if the following congruences are true or false:

- $25 \equiv 1 \pmod{6}$
- $100 \equiv 20 \pmod{10}$
- $52 \equiv 0 \pmod{13}$
- $75 \equiv 7 \pmod{5}$

Recall: $x \bmod y$ equals the remainder when dividing x by y.

Find the following values:

(a) $34 \bmod 5 =$ ________
(b) $78 \bmod 11 =$ ________
(c) $13 \bmod 15 =$ ________
(d) $12 \bmod 2 =$ ________

Types of Errors in Identification Numbers

- **Single digit error:** Replacing one digit with another (e.g., ac entered instead of ab)
- **Adjacent transposition error:** Swapping two adjacent digits (e.g., ba entered instead of ab)
- **Jump transposition error:** Transposing a sequence of digits (e.g., cba entered instead of abc)

Note: Some digits in the UPC code are multiplied by 3. Those digits have a *weight* of 3.

Check Digits in 6-Digit Codes

A code $a_1a_2a_3a_4a_5a_6$ uses the last digit a_6 as a **check digit**. The check digit is computed using the formula:

$$a_6 = (3a_1 + a_2 + 3a_3 + a_4 + 5a_5) \bmod 10$$

This formula helps detect common data entry errors.

(a) What is the check digit for the code `23714_`?

We are given: $a_1 = 2,\ a_2 = 3,\ a_3 = 7,\ a_4 = 1,\ a_5 = 4$

$$\begin{aligned} a_6 &= (3(2) + 3 + 3(7) + 1 + 5(4)) \bmod 10 \\ &= (6 + 3 + 21 + 1 + 20) \bmod 10 \\ &= 51 \bmod 10 = 1 \end{aligned}$$

Answer: The full code is `237141`.

(b) Find the value of the missing digit x in the code `46_782`

We are given:

$$a_1 = 4, \quad a_2 = 6, \quad a_3 = x, \quad a_4 = 7, \quad a_5 = 8, \quad a_6 = 2$$

Apply the check digit formula:

$$a_6 = (3a_1 + a_2 + 3a_3 + a_4 + 5a_5) \bmod 10$$

Substitute the known values:

$$2 = (3(4) + 6 + 3x + 7 + 5(8)) \bmod 10$$

$$2 = (12 + 6 + 3x + 7 + 40) \bmod 10 = (65 + 3x) \bmod 10$$

Solve:

$$65 + 3x \equiv 2 \pmod{10} \Rightarrow 3x \equiv -63 \equiv 7 \pmod{10}$$

To solve $3x \equiv 7 \pmod{10}$, multiply both sides by the modular inverse of 3 mod 10.

- The inverse of 3 mod 10 is 7, since $3 \times 7 = 21 \equiv 1 \pmod{10}$
- Multiply both sides: $x \equiv 7 \times 7 = 49 \equiv 9 \pmod{10}$

Answer: $x = 9$, so the complete code is `469782`.

(c) Will this code detect an error if the first digit is entered incorrectly?

Yes, because a_1 is multiplied by 3 in the check digit formula. A change in a_1 by 1 would change the weighted sum by 3, which likely changes the check digit.

Answer: Yes, most errors in a_1 will be detected by this check digit method.

Check Digit System: $a_3 = a_1 + 4a_2 \mod 9$

We are given a 3-digit code of the form $a_1a_2a_3$, where a_3 is a **check digit** computed by the formula:

$$a_3 = (a_1 + 4a_2) \mod 9$$

This method aims to detect errors in data entry.

(a) Will this check digit find all transposition errors?

A **transposition error** occurs when two adjacent digits are swapped. We are concerned with the case where a_1 and a_2 are swapped, producing the incorrect code $a_2 a_1 a_3'$.

$$\text{Original check digit: } a_3 = a_1 + 4a_2 \mod 9$$
$$\text{Transposed check digit: } a_3' = a_2 + 4a_1 \mod 9$$

We now compare:

$$a_3 - a_3' = (a_1 + 4a_2) - (a_2 + 4a_1) = -3a_1 + 3a_2 = 3(a_2 - a_1)$$

- If $a_2 \neq a_1$, then $a_3 \neq a_3'$ — the error will be detected.
- If $a_2 = a_1$, then the transposition makes no change — but no error was actually made.

Answer: Yes, this check digit detects all adjacent transposition errors of a_1 and a_2.

(b) Will this check digit find all single-digit errors in the first position?

Let the incorrect first digit be $a_1' \neq a_1$.
Original check digit:

$$a_3 = a_1 + 4a_2 \mod 9$$

Incorrect check digit (with error in a_1):

$$a_3' = a_1' + 4a_2 \mod 9$$

Then:

$$a_3' - a_3 = a_1' - a_1$$

Since any change in a_1 changes a_3, the error is detected.
Answer: Yes, all single-digit errors in the first position are detected.

(c) Will this check digit find all single-digit errors in the second position?

Let the incorrect second digit be $a_2' \neq a_2$.
Original check digit:

$$a_3 = a_1 + 4a_2 \mod 9$$

Incorrect check digit (with error in a_2):

$$a_3' = a_1 + 4a_2' \mod 9$$

Then:

$$a_3' - a_3 = 4(a_2' - a_2)$$

Again, any change in a_2 results in a change in a_3, so the error is detected.
Answer: Yes, all single-digit errors in the second position are detected.

Check Digit System: $a_4 = 7a_1 + 2a_2 + 5a_3 \mod 9$

We are given a 4-digit code of the form $a_1a_2a_3a_4$, where a_4 is a **check digit** computed by the formula:

$$a_4 = (7a_1 + 2a_2 + 5a_3) \mod 9$$

(a) Determine the value of x in the code 2x45, given that the check digit is valid.

We are given:

$$a_1 = 2, \quad a_2 = x, \quad a_3 = 4, \quad a_4 = 5$$

Plug into the formula:

$$a_4 = (7(2) + 2x + 5(4)) \mod 9 = (14 + 2x + 20) \mod 9 = (34 + 2x) \mod 9 = 5$$

Now solve:

$$34 + 2x \equiv 5 \pmod 9$$

Reduce 34 mod 9:

$$34 \mod 9 = 7 \Rightarrow 7 + 2x \equiv 5 \pmod 9 \Rightarrow 2x \equiv -2 \equiv 7 \pmod 9$$

Now solve:

$$2x \equiv 7 \pmod 9$$

Multiply both sides by the inverse of 2 mod 9. Since $2 \times 5 = 10 \equiv 1 \mod 9$, the inverse is 5.

$$x \equiv 5 \cdot 7 \equiv 35 \equiv 8 \pmod 9$$

Answer: $x = 8$

(b) Will this check digit detect all single-digit errors in the third position?

Let's suppose a_3 is changed to a_3', and all other digits are correct. Then:

$$\begin{aligned} \text{Correct:} \quad & a_4 = (7a_1 + 2a_2 + 5a_3) \mod 9 \\ \text{Incorrect:} \quad & a_4' = (7a_1 + 2a_2 + 5a_3') \mod 9 \end{aligned}$$

So:

$$a_4' - a_4 = 5(a_3' - a_3)$$

- If $a_3' \neq a_3$, then $a_4' \neq a_4$ — the error will be detected.
- If $a_3' = a_3$, then no error was made.

Answer: Yes, this check digit detects all single-digit errors in the third position.

(c) Will this check digit detect all transposition errors in the second and third positions?

Assume a transposition of a_2 and a_3. The correct check digit is:

$$a_4 = (7a_1 + 2a_2 + 5a_3) \mod 9$$

After transposition:

$$a'_4 = (7a_1 + 2a_3 + 5a_2) \mod 9$$

Compare:

$$a'_4 - a_4 = (2a_3 + 5a_2) - (2a_2 + 5a_3) = 3(a_2 - a_3)$$

- If $a_2 \neq a_3$, then $a'_4 \neq a_4$ — the error will be detected.
- If $a_2 = a_3$, then no change occurs — but no error was made.

Answer: Yes, this check digit detects all transposition errors between the second and third positions.

End of Lecture #10