Introduction to Mathematical Modeling

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Apportionment Problems

Introduction

An **apportionment problem** arises when a set of fractional allocations must be rounded to integers while maintaining the sum of the original fractions. The rounding procedure used in such problems is called an **apportionment method**.

Notation and Rounding

- Round q to the nearest integer: $[q]$, rounding half integers **up**.
- Round q **down**: $\lfloor q \rfloor$
- Round q **up**: $\lceil q \rceil$

Standard Divisor and Quota

Definition 1 (Standard Divisor)**.** *If p is the total population and h is the house size, the **standard divisor** s is*

$$s = \frac{p}{h}.$$

Definition 2 (Quota)**.** *A group's **quota** q_i is the group's population p_i divided by the standard divisor:*

$$q_i = \frac{p_i}{s}.$$

Hamilton Method

The Hamilton method is a classic apportionment procedure:

1. Round each quota q_i **down** to get initial seats.
2. Calculate the number of seats left to assign:

$$h - \sum_i \lfloor q_i \rfloor$$

3. Assign the remaining seats to groups with the **largest fractional parts** $q_i - \lfloor q_i \rfloor$ until all seats are distributed.

Example 1

Suppose a population of $p = 1000$ is divided among 3 groups:

$$p_1 = 450, \quad p_2 = 350, \quad p_3 = 200$$

and the house size is $h = 10$.

1. Compute the standard divisor:

$$s = \frac{1000}{10} = 100$$

2. Compute each quota:

$$q_1 = \frac{450}{100} = 4.5, \quad q_2 = \frac{350}{100} = 3.5, \quad q_3 = \frac{200}{100} = 2$$

3. Round down:

$$\lfloor q_1 \rfloor = 4, \quad \lfloor q_2 \rfloor = 3, \quad \lfloor q_3 \rfloor = 2$$

   Total assigned: $4 + 3 + 2 = 9$. One seat remains.

4. Assign the remaining seat to the group with the largest fractional part. Here:

$$q_1 - \lfloor q_1 \rfloor = 0.5, \quad q_2 - \lfloor q_2 \rfloor = 0.5, \quad q_3 - \lfloor q_3 \rfloor = 0$$

   If ties are broken arbitrarily, assign the extra seat to group 1.

Final apportionment:

$$\text{Group 1: 5 seats, Group 2: 3 seats, Group 3: 2 seats}$$

Example 2: Apportioning Silver Coins Using Hamilton Method

Suppose 36 silver coins are to be distributed among Doris, Mildred, and Henrietta based on their payments:

$$\text{Doris: } 5900, \quad \text{Mildred: } 7600, \quad \text{Henrietta: } 1400$$

1. Compute the total payment:

$$p = 5900 + 7600 + 1400 = 14900$$

2. Compute the standard divisor:

$$s = \frac{p}{h} = \frac{14900}{36} \approx 413.89$$

3. Compute each group's quota:

$$q_{\text{Doris}} = \frac{5900}{413.89} \approx 14.25$$

$$q_{\text{Mildred}} = \frac{7600}{413.89} \approx 18.37$$

$$q_{\text{Henrietta}} = \frac{1400}{413.89} \approx 3.38$$

4. Round down each quota:

$$\lfloor q_{\text{Doris}} \rfloor = 14, \quad \lfloor q_{\text{Mildred}} \rfloor = 18, \quad \lfloor q_{\text{Henrietta}} \rfloor = 3$$

   Total assigned: $14 + 18 + 3 = 35$ seats. One seat remains.

5. Assign the remaining seat to the group with the largest fractional part:

$$0.25 \text{ (Doris)}, \quad 0.37 \text{ (Mildred)}, \quad 0.38 \text{ (Henrietta)}$$

   The largest fractional part is Henrietta's 0.38. So she receives the extra seat.

Final Apportionment: Doris: 14, Mildred: 18, Henrietta: 4

Example: Adjusting Apportionment for 37 Coins

Suppose the same payments are used (Doris: 5900, Mildred: 7600, Henrietta: 1400), but now there are 37 coins instead of 36.

1. Compute the new standard divisor:

$$s = \frac{p}{h} = \frac{14900}{37} \approx 402.70$$

2. Compute each group's quota:

$$q_{\text{Doris}} = \frac{5900}{402.70} \approx 14.65$$

$$q_{\text{Mildred}} = \frac{7600}{402.70} \approx 18.88$$

$$q_{\text{Henrietta}} = \frac{1400}{402.70} \approx 3.48$$

3. Round down each quota:

$$\lfloor q_{\text{Doris}} \rfloor = 14, \quad \lfloor q_{\text{Mildred}} \rfloor = 18, \quad \lfloor q_{\text{Henrietta}} \rfloor = 3$$

   Total assigned: $14 + 18 + 3 = 35$ coins. Two coins remain.

4. Assign the remaining coins to the groups with the largest fractional parts:

$$q_{\text{Doris}} - \lfloor q_{\text{Doris}} \rfloor = 0.65$$

$$q_{\text{Mildred}} - \lfloor q_{\text{Mildred}} \rfloor = 0.88$$

$$q_{\text{Henrietta}} - \lfloor q_{\text{Henrietta}} \rfloor = 0.48$$

   The two largest fractional parts are Mildred (0.88) and Doris (0.65). Assign one coin to each.

Final Apportionment: Doris: 15, Mildred: 19, Henrietta: 3

Practice Problems

1. A population of 1200 is divided among 4 groups: $500, 300, 250, 150$. The house size is 12. Use the Hamilton method to allocate seats.

2. Explain what happens if all fractional parts are equal when assigning remaining seats.

3. Compare Hamilton method with simple rounding of quotas. Which can cause discrepancies?

End of Lecture #13