Introduction to Mathematical Modeling

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Paradoxes in Apportionment and Differences

Paradoxes in Apportionment

Definition 1. *A* ***paradox*** *is a statement that is seemingly contradictory or opposed to common sense, yet may be true.*

Some well-known apportionment paradoxes:

- **Alabama Paradox:** A state loses a seat as a result of an increase in the house size.
- **New States Paradox:** Adding a new state or increasing the total number of seats causes a shift in the apportionment of existing states.
- **Population Paradox:** With a fixed number of seats, a state loses a seat to another state even though its population grew faster.

Absolute and Relative Differences

Definition 2. *For two numbers A and B where $A > B$:*

- *The* ***absolute difference*** *is:*
$$A - B$$
- *The* ***relative difference*** *(percentage) is:*
$$\frac{A - B}{B} \times 100\%$$

Examples

(a) $A = 2, B = 1$ Absolute difference: $2 - 1 = 1$ Relative difference: $\frac{2-1}{1} \times 100\% = 100\%$

(b) $A = 11, B = 10$ Absolute difference: $11 - 10 = 1$ Relative difference: $\frac{11-10}{10} \times 100\% = 10\%$

(c) $A = 2001, B = 2000$ Absolute difference: $2001 - 2000 = 1$ Relative difference: $\frac{2001-2000}{2000} \times 100\% = 0.05\%$

Example: Apportioning a 100-Member Advisory Council

Suppose a county has four districts: North, South, East, and West, with populations as follows:

North: 12000, South: 15000, East: 9000, West: 13000

We want to apportion a 100-member advisory council using the **Hamilton Method**.

1. Compute the total population:

$$p = 12000 + 15000 + 9000 + 13000 = 49000$$

2. Compute the standard divisor:

$$s = \frac{p}{h} = \frac{49000}{100} = 490$$

3. Compute each district's quota:

$$q_{\text{North}} = \frac{12000}{490} \approx 24.49$$

$$q_{\text{South}} = \frac{15000}{490} \approx 30.61$$

$$q_{\text{East}} = \frac{9000}{490} \approx 18.37$$

$$q_{\text{West}} = \frac{13000}{490} \approx 26.53$$

4. Round down each quota:

$$\lfloor q_{\text{North}} \rfloor = 24, \quad \lfloor q_{\text{South}} \rfloor = 30, \quad \lfloor q_{\text{East}} \rfloor = 18, \quad \lfloor q_{\text{West}} \rfloor = 26$$

Total assigned: $24 + 30 + 18 + 26 = 98$ seats. Two seats remain.

5. Assign the remaining seats to the districts with the largest fractional parts:

North: 0.49, South: 0.61, East: 0.37, West: 0.53

The two largest fractional parts are South (0.61) and West (0.53). Assign one extra seat to each.

Final Apportionment: North: 24, South: 31, East: 18, West: 27

Example: Reapportioning 100 Seats Ten Years Later

Step 1: Total population:

$$p = 14000 + 16000 + 12000 + 13000 = 55000$$

Step 2: Standard divisor:

$$s = \frac{p}{h} = \frac{55000}{100} = 550$$

Step 3: Compute quotas for each district:

$$q_{\text{North}} = \frac{14000}{550} \approx 25.45$$

$$q_{\text{South}} = \frac{16000}{550} \approx 29.09$$

$$q_{\text{East}} = \frac{12000}{550} \approx 21.82$$

$$q_{\text{West}} = \frac{13000}{550} \approx 23.64$$

Step 4: Round down each quota:

$$\lfloor q_{\text{North}} \rfloor = 25, \quad \lfloor q_{\text{South}} \rfloor = 29, \quad \lfloor q_{\text{East}} \rfloor = 21, \quad \lfloor q_{\text{West}} \rfloor = 23$$

Step 5: Seats assigned so far:

$$25 + 29 + 21 + 23 = 98$$

Two seats remain.

Step 6: Assign remaining seats to districts with largest fractional parts: Fractional parts:

$$\text{North: } 0.45, \quad \text{South: } 0.09, \quad \text{East: } 0.82, \quad \text{West: } 0.64$$

The two largest fractional parts are East (0.82) and West (0.64). Assign one extra seat to each.

$$\text{Final Apportionment: North: 25, South: 29, East: 22, West: 24}$$

Step 7: Check for Paradox

Compare with previous apportionment (North: 24, South: 31, East: 18, West: 27):

- North: $24 \to 25$ (gain)

- South: $31 \to 29$ (loss)
- East: $18 \to 22$ (gain)
- West: $27 \to 24$ (loss)

Even though South's population increased, it lost seats. This is an example of the **population paradox**.

Example: Adding a New State (Plasma)

Step 1: Compute total population including Plasma:

$$p = 14000 + 16000 + 12000 + 13000 + 38240 = 93240$$

Step 2: Compute standard divisor:

$$s = \frac{p}{h} = \frac{93240}{100} = 932.4$$

Step 3: Compute quotas:

$$q_{\text{North}} = \frac{14000}{932.4} \approx 15.01$$
$$q_{\text{South}} = \frac{16000}{932.4} \approx 17.16$$
$$q_{\text{East}} = \frac{12000}{932.4} \approx 12.87$$
$$q_{\text{West}} = \frac{13000}{932.4} \approx 13.94$$
$$q_{\text{Plasma}} = \frac{38240}{932.4} \approx 41.01$$

Step 4: Round down each quota:

$$\lfloor q_{\text{North}} \rfloor = 15, \quad \lfloor q_{\text{South}} \rfloor = 17, \quad \lfloor q_{\text{East}} \rfloor = 12, \quad \lfloor q_{\text{West}} \rfloor = 13, \quad \lfloor q_{\text{Plasma}} \rfloor = 41$$

Seats assigned so far:

$$15 + 17 + 12 + 13 + 41 = 98$$

Two seats remain.

Step 5: Assign remaining seats to districts with largest fractional parts: Fractional parts:

$$\text{North: } 0.01, \quad \text{South: } 0.16, \quad \text{East: } 0.87, \quad \text{West: } 0.94, \quad \text{Plasma: } 0.01$$

The two largest fractional parts are West (0.94) and East (0.87). Assign one extra seat to each.

$$\text{Final Apportionment: North: 15, South: 17, East: 13, West: 14, Plasma: 41}$$

Step 6: Observation

Adding Plasma caused a change in the distribution of the other districts' seats compared to before. This can illustrate the **New States Paradox**, where introducing a new state shifts seats among existing states.

Divisor Methods and the Jefferson Method

Divisor Methods

The **standard divisor**, s, represents the average population per seat. Apportionment can also be done using a specific **adjusted divisor**, d, chosen to match a target total number of seats.

Definition 3 (Divisor Method)**.** *A* divisor method *of apportionment determines each state's quota by*

$$q_i = \frac{p_i}{d},$$

where p_i is the population of state i, and then rounds the resulting quota according to a specific rounding rule. A critical divisor *is a value of d that produces a quota for each population such that the total number of seats is exactly correct.*

Different divisor methods use different rounding rules. The Jefferson method is one of the earliest and simplest divisor methods.

The Jefferson Method

1. Compute the standard divisor s and standard quotas $q_i = \frac{p_i}{s}$. Round each quota **down** (floor function).

2. If the total number of seats is not correct, compute a new divisor for each state that corresponds to giving one more seat:

$$d_i = \frac{p_i}{q_i + 1}.$$

3. Assign a seat to the state with the largest d_i. Repeat step 2 until the total number of seats matches the desired total.

4. The final adjusted divisor d is the exact value of the last divisor found.

Example 1: Apportioning 36 Silver Coins

Problem: Apportion 36 coins to Doris, Mildred, and Henrietta with populations/payments:

$$\text{Doris: } 5900, \quad \text{Mildred: } 7600, \quad \text{Henrietta: } 1400$$

Step 1: Standard Divisor

$$s = \frac{5900 + 7600 + 1400}{36} = \frac{14900}{36} \approx 413.89$$

Step 2: Standard Quotas (rounded down)

$$q_{\text{Doris}} = \left\lfloor \frac{5900}{413.89} \right\rfloor = 14, \quad q_{\text{Mildred}} = \left\lfloor \frac{7600}{413.89} \right\rfloor = 18, \quad q_{\text{Henrietta}} = \left\lfloor \frac{1400}{413.89} \right\rfloor = 3$$

Step 3: Total Seats Assigned

$$14 + 18 + 3 = 35 < 36$$

One more coin must be assigned. Compute adjusted divisors for giving one more seat:

$$d_{\text{Doris}} = \frac{5900}{15} \approx 393.33, \quad d_{\text{Mildred}} = \frac{7600}{19} \approx 400.00, \quad d_{\text{Henrietta}} = \frac{1400}{4} = 350$$

Step 4: Assign the extra coin The largest d_i is Mildred, so she receives 1 extra coin.

Final Apportionment:

$$\text{Doris: } 14, \text{ Mildred: } 19, \text{ Henrietta: } 3$$

Comment: Jefferson's method slightly favors larger populations because it always rounds down quotas.

Example 2: When the Bag Has 37 Coins

If there are 37 coins, the same procedure is followed. The adjusted divisor changes slightly, and additional coins are distributed to states with largest d_i. Jefferson's method may give slightly different allocations than Hamilton's method, again favoring larger states.

Example 3: Assigning Teachers to Art Classes

A school has 4 art classes with enrollments:

$$\text{Ceramics: } 785, \text{ Painting: } 152, \text{ Dance: } 160, \text{ Theatre: } 95$$

Ten new teachers are to be assigned using Jefferson's method. Compute the standard divisor $s = \frac{785+152+160+95}{10} = 1192/10 = 119.2$, compute quotas, round down, and assign remaining teachers to classes with largest adjusted divisors.

Comment: Jefferson's method favors larger classes (Ceramics, Dance) over smaller ones (Theatre).

The Quota Rule and Paradoxes

The **Quota Rule** states that the number of seats assigned to each unit should equal its standard quota rounded up or down. Balinski and Young proved that **no apportionment method that satisfies the quota rule is completely free of paradoxes**. Jefferson's method may violate the quota rule but avoids the Alabama paradox, while Hamilton's method satisfies the quota rule but may produce paradoxes.

The Adams and Webster Methods

The Adams Method

The Adams method is a **divisor method** similar to Jefferson's, but it tends to favor smaller populations.

1. Compute the standard divisor $s = \frac{\text{total population}}{\text{total seats}}$ and quotas
$$q_i = \frac{p_i}{s}.$$
Round each quota **up** (ceiling function):
$$N_i = \lceil q_i \rceil.$$
2. If the total number of seats assigned is not correct, compute a new divisor for each state corresponding to giving one fewer seat:
$$d_i = \frac{p_i}{N_i - 1}.$$
3. Remove a seat from the state with the **smallest** d_i. Repeat steps 2–3 until the total number of seats matches the desired total.
4. The final adjusted divisor d is the exact value of the last divisor found.

Comment: Adams' method favors smaller populations because it rounds quotas up.

The Webster Method

The Webster method uses **standard rounding** (to the nearest integer) instead of always rounding up or down.

1. Compute the standard divisor $s = \frac{\text{total population}}{\text{total seats}}$ and quotas
$$q_i = \frac{p_i}{s}.$$
Round each quota to the nearest integer:
$$N_i = [q_i]\,.$$

2. If the total number of seats assigned equals the desired total, stop.

3. If the total number of seats is **too few**, compute a critical divisor d^+ for each state:

$$d_i^+ = \frac{p_i}{N_i + 1}.$$

   Assign the next seat to the state with the **largest** d_i^+. Repeat until the total matches the desired number of seats.

4. If the total number of seats is **too many**, compute a critical divisor d^- for each state:

$$d_i^- = \frac{p_i}{N_i - 1}.$$

   Remove a seat from the state with the **smallest** d_i^-. Repeat until the total matches the desired number of seats.

Comment: Webster's method is the most balanced, tending to minimize bias between small and large populations. It also generally satisfies the **quota rule** more often than Jefferson or Adams methods.

Apportionment using Adams and Webster Methods

We are asked to apportion a house of size $H = 16$ among four regions with populations:

$$\text{Beach: } 28{,}204, \quad \text{Forest: } 11{,}267, \quad \text{Plains: } 4{,}203, \quad \text{Swamp: } 1{,}462$$

Step 1: Standard Divisor

The total population is

$$P = 28{,}204 + 11{,}267 + 4{,}203 + 1{,}462 = 45{,}136$$

The standard divisor is

$$s = \frac{P}{H} = \frac{45{,}136}{16} \approx 2{,}821$$

Step 2: Quotas

The quota for each region is

$$q_i = \frac{p_i}{s}$$

$$q_{\text{Beach}} = \frac{28{,}204}{2{,}821} \approx 10.0$$
$$q_{\text{Forest}} = \frac{11{,}267}{2{,}821} \approx 4.0$$
$$q_{\text{Plains}} = \frac{4{,}203}{2{,}821} \approx 1.49$$
$$q_{\text{Swamp}} = \frac{1{,}462}{2{,}821} \approx 0.52$$

Step 3: Adams Method

Adams method rounds **all quotas up**:

$$N_{\text{Beach}} = \lceil 10.0 \rceil = 10$$
$$N_{\text{Forest}} = \lceil 4.0 \rceil = 4$$
$$N_{\text{Plains}} = \lceil 1.49 \rceil = 2$$
$$N_{\text{Swamp}} = \lceil 0.52 \rceil = 1$$

Total seats: $10+4+2+1 = 17 > 16$, so we must remove 1 seat. We remove a seat from the region with the smallest critical divisor:

$$d_i = \frac{p_i}{N_i - 1} \quad (N_i > 1)$$

$$d_{\text{Beach}} = \frac{28{,}204}{10-1} \approx 3{,}134.9$$
$$d_{\text{Forest}} = \frac{11{,}267}{4-1} \approx 3{,}755.7$$
$$d_{\text{Plains}} = \frac{4{,}203}{2-1} = 4{,}203$$

Beach has the smallest d_i, so we remove 1 seat from Beach.

Comment: Adams method tends to favor smaller regions, giving Plains 2 seats instead of rounding down.

Step 4: Webster Method

Webster method rounds quotas to the **nearest integer**:

Region	Quota	Adams Seats
Beach	10.0	9
Forest	4.0	4
Plains	1.49	2
Swamp	0.52	1
Total	–	16

$$N_{\text{Beach}} = [10.0] = 10$$
$$N_{\text{Forest}} = [4.0] = 4$$
$$N_{\text{Plains}} = [1.49] = 1$$
$$N_{\text{Swamp}} = [0.52] = 1$$

Total seats: $10 + 4 + 1 + 1 = 16$ (matches house size).

Region	Quota	Webster Seats
Beach	10.0	10
Forest	4.0	4
Plains	1.49	1
Swamp	0.52	1
Total	–	16

Comment: Webster method gives a more balanced and proportional distribution, slightly favoring neither small nor large regions excessively.

End of Lecture #14