MATH 108: Elementary Probability and Statistics

Ramapo College of New Jersey

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Semester: Fall 2025 Date: October 27, 2025

Uniform and Normal Probability Distributions

Continuous Random Variable:
A *continuous random variable* is one that can take on **any value within a range** of numbers, not just specific separate values.

- A **discrete** random variable has countable outcomes (e.g., rolling a die: 1, 2, 3, 4, 5, 6).
- A **continuous** random variable can take any value along a continuum (e.g., 0 to 30 minutes late, including 5.2, 5.25, or 5.257 minutes).

For continuous variables, the probability of getting an **exact value** (like exactly 5 minutes late) is **zero**. Instead, we consider the probability of being within a **range**, such as between 5 and 10 minutes.

Probability Density Function (PDF)

A *probability density function* is an equation that describes how probabilities are distributed for a continuous random variable.

A valid pdf must satisfy:

1. The **total area** under the graph equals 1 (all possible outcomes together have probability 1).
2. The **height** of the graph (the pdf value) is always greater than or equal to 0.

Example

Let

$$X = \text{number of minutes your friend is late.}$$

Your friend could arrive anytime from 0 to 30 minutes late, with all times equally likely.
Thus,

$$X \sim \text{Uniform}(0, 30).$$

Since all values are equally likely, the probability of being between two times depends only on the **length of the interval**:

$$P(x_1 \le X \le x_2) = \frac{x_2 - x_1}{30}.$$

Example Calculations:

(a) Probability that your friend is between 10 and 20 minutes late:

$$P(10 \le X \le 20) = \frac{20 - 10}{30} = \frac{1}{3} \approx 0.333$$

(b) It is 10 a.m. There is a 20% chance your friend will arrive within the next t minutes:

$$P(0 \le X \le t) = 0.20 \Rightarrow \frac{t}{30} = 0.20 \Rightarrow t = 6$$

So there is a 20% chance your friend will arrive within the next **6 minutes**.

Practice Problems: Uniform Distribution

Problems 11–14: Friend Arrival Time (Uniform Distribution, $0 \leq X \leq 30$ minutes)

(11) (a) Find the probability that your friend is between 5 and 10 minutes late.
(b) It is 10 a.m. There is a 40% probability your friend will arrive within the next ____ minutes.

(12) (a) Find the probability that your friend is between 15 and 25 minutes late.
(b) It is 10 a.m. There is a 90% probability your friend will arrive within the next ____ minutes.

(13) Find the probability that your friend is at least 20 minutes late.

(14) Find the probability that your friend is no more than 5 minutes late.

Problems 15–16: Uniform Distribution Examples

15. The random-number generator on calculators randomly generates a number between 0 and 1. Let X be the number generated; X follows a uniform probability distribution.
    (a) Draw the graph of the uniform density function.
    (b) What is the probability of generating a number between 0 and 0.2?
    (c) What is the probability of generating a number between 0.25 and 0.6?
    (d) What is the probability of generating a number greater than 0.95?
    (e) Use your calculator or statistical software to randomly generate 200 numbers between 0 and 1. What proportion of the numbers are between 0 and 0.2? Compare the result with part (b).
16. The reaction time X (in minutes) of a chemical process follows a uniform probability distribution with $5 \leq X \leq 10$.
    (a) Draw the graph of the density curve.
    (b) What is the probability that the reaction time is between 6 and 8 minutes?
    (c) What is the probability that the reaction time is between 5 and 8 minutes?
    (d) What is the probability that the reaction time is less than 6 minutes?

Solutions: Uniform Distribution Problems

Problems 11–14: Friend Arrival Time ($X \sim$ Uniform$(0, 30)$)

(11) (a) $P(5 \leq X \leq 10) = \dfrac{10-5}{30} = 0.1667 \approx 16.7\%$

(b) $0.4 = \dfrac{t-0}{30} \implies t = 12$ minutes

(12) (a) $P(15 \leq X \leq 25) = \dfrac{25-15}{30} = 0.3333 \approx 33.3\%$

(b) $0.9 = \dfrac{t-0}{30} \implies t = 27$ minutes

(13) $P(X \geq 20) = \dfrac{30-20}{30} = 0.3333 \approx 33.3\%$

(14) $P(X \leq 5) = \dfrac{5-0}{30} = 0.1667 \approx 16.7\%$

Problems 15: Random Number Generator ($X \sim$ Uniform$(0, 1)$)

(b) $P(0 \leq X \leq 0.2) = 0.2 = 20\%$

(c) $P(0.25 \le X \le 0.6) = 0.6 - 0.25 = 0.35 = 35\%$

(d) $P(X > 0.95) = 1 - 0.95 = 0.05 = 5\%$

(e) Simulation: Generating 200 numbers, expected proportion in $[0, 0.2]$ is ≈ 0.2; actual proportion may vary slightly.

Problem 16: Reaction Time ($X \sim$ Uniform$(5, 10)$)

(b) $P(6 \le X \le 8) = \dfrac{8-6}{10-5} = \dfrac{2}{5} = 0.4 = 40\%$

(c) $P(5 \le X \le 8) = \dfrac{8-5}{5} = 0.6 = 60\%$

(d) $P(X \le 6) = \dfrac{6-5}{5} = 0.2 = 20\%$

Practice Problems: Probability Density Functions (PDFs)

17. The probability density function of a continuous random variable X is given by:

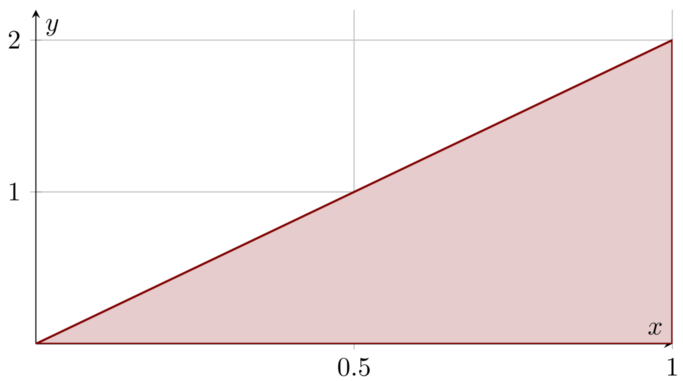
$$y = 2x, \quad \text{for } 0 \le x \le 1$$

(a) Draw the graph of the density curve for the continuous random variable.
(b) Verify that it is a valid probability density function.
(c) Find the probability that $X < \frac{1}{2}$.
(d) Find the probability that $X > \frac{1}{2}$.
(e) Find the probability that $X = \frac{1}{2}$.

Solutions: Probability Density Function Problem

(17) (a) The graph of $y = 2x$ for $0 \le x \le 1$ is a straight line starting at $(0, 0)$ and ending at $(1, 2)$. It forms a right triangle under the line and above the x-axis.

PDF: $y = 2x,\ 0 \le x \le 1$



(b) The area under the curve represents the total probability. The shape is a right triangle with base 1 and height 2:

$$\text{Area} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(1)(2) = 1$$

Since the total area equals 1 and the graph never goes below the x-axis, this is a valid probability density function.

(c) To find $P(X < \frac{1}{2})$, look at the smaller triangle from $x = 0$ to $x = 0.5$. The base is 0.5 and the height at $x = 0.5$ is $y = 2(0.5) = 1$.

$$\text{Area} = \tfrac{1}{2}(0.5)(1) = 0.25$$

So, $P(X < \frac{1}{2}) = 0.25$.

(d) The total probability is 1, so:

$$P(X > \tfrac{1}{2}) = 1 - 0.25 = 0.75$$

(e) For a continuous random variable, the probability at a single point is zero:

$$P(X = \tfrac{1}{2}) = 0$$

18. The probability density function of a continuous random variable X is given by:

$$y = \tfrac{1}{2} - \tfrac{1}{2}x, \quad \text{for } -1 \leq x \leq 1$$

(a) Draw the graph of the density curve for the continuous random variable.
(b) Verify that it is a valid probability density function.
(c) Find the probability that $X > \frac{1}{2}$.
(d) Find the probability that $X < \frac{1}{2}$.
(e) Find the probability that $X = \frac{1}{2}$.

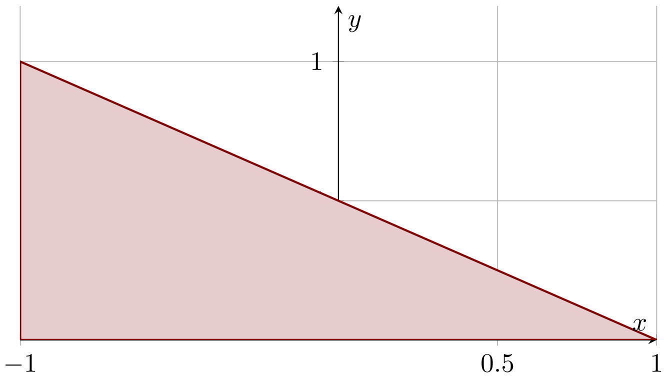
Solution: Probability Density Function Problem

(18) (a) The graph of $y = \frac{1}{2} - \frac{1}{2}x$ is a straight line that:

- starts at $(x, y) = (-1, 1)$,
- ends at $(x, y) = (1, 0)$,
- slopes downward from left to right.

The shaded area under this line represents the total probability.

PDF: $y = \frac{1}{2} - \frac{1}{2}x$, $-1 \leq x \leq 1$



(b) There are two conditions that must be satisfied:

1. The graph must be above the x-axis. From the picture, $y \geq 0$ for all $-1 \leq x \leq 1$.
2. The total area under the curve must equal 1. The shape is a right triangle:

$$A = \tfrac{1}{2}(\text{base})(\text{height})$$

The base extends from $x = -1$ to $x = 1$, so $b = 2$. The height is $h = 1$ (from $y = 0$ to $y = 1$).

$$A = \tfrac{1}{2}(2)(1) = 1$$

Therefore, it is a valid probability density function.

(c) To find $P(X > \frac{1}{2})$, look at the small red triangle on the right side of the graph.
- The base runs from $x = \frac{1}{2}$ to $x = 1$, so $b = 1 - 0.5 = 0.5$. - The height at $x = \frac{1}{2}$ is:

$$y = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} = 0.25$$

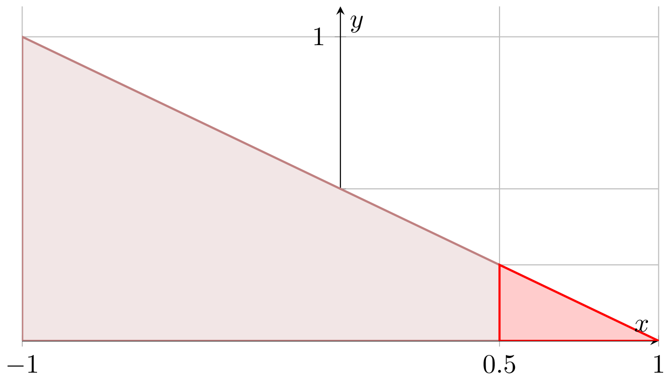
So the height of the triangle is $h = 0.25$.
The area (probability) is:

$$A = \frac{1}{2}(b)(h) = \frac{1}{2}(0.5)(0.25) = 0.0625$$

Thus,

$$P(X > \tfrac{1}{2}) = 0.0625 = \frac{1}{16}$$

Shaded Region: $P(X > \frac{1}{2})$



(d)

$$P(X < \tfrac{1}{2}) = 1 - P(X > \tfrac{1}{2}) = 1 - 0.0625 = 0.9375 = \frac{15}{16}$$

(e) For a continuous random variable, the probability at a single point is always zero:

$$P(X = \tfrac{1}{2}) = 0$$

Normal Distribution

A continuous random variable is said to be *normally distributed* if its relative frequency histogram forms the shape of a **normal curve**, also called a **bell curve**.

- The curve is **symmetric** about the mean, μ.
- The standard deviation, σ, determines the spread of the curve.
- The points at $x = \mu - \sigma$ and $x = \mu + \sigma$ are called **inflection points**. These are points where the curve changes from concave downward to concave upward (or vice versa):
  - To the left of $\mu - \sigma$ and to the right of $\mu + \sigma$, the curve bends upward.
  - Between $\mu - \sigma$ and $\mu + \sigma$, the curve bends downward.
- The highest point of the curve occurs at $x = \mu$, the mean of the distribution.

Properties of the Normal Density Curve

The normal probability density function satisfies all requirements of a probability distribution. Its key properties are:

1. **Symmetry:** The curve is symmetric about its mean, μ.
2. **Single Peak:** Since mean = median = mode, the normal curve has a single peak at $x = \mu$.

3. **Inflection Points:** The curve changes concavity at $x = \mu - \sigma$ and $x = \mu + \sigma$.
4. **Total Area:** The total area under the curve equals 1.
5. **Equal Halves:** The area under the curve to the right of the mean equals the area to the left of the mean, each equal to $\frac{1}{2}$.
6. **Asymptotic Behavior:** As $x \to \infty$ or $x \to -\infty$, the curve approaches, but never touches, the horizontal axis.
7. Approximately 68% of the area under the curve lies within one standard deviation of the mean:
$$\mu - \sigma \leq X \leq \mu + \sigma$$
8. Approximately 95% of the area lies within two standard deviations of the mean:
$$\mu - 2\sigma \leq X \leq \mu + 2\sigma$$
9. Approximately 99.7% of the area lies within three standard deviations of the mean:
$$\mu - 3\sigma \leq X \leq \mu + 3\sigma$$

Definition

Area under a Normal Curve:
Suppose a random variable X is normally distributed with mean μ and standard deviation σ. The area under the normal curve for any interval of X represents either:

- The **proportion of the population** with values in that interval, or
- The **probability** that a randomly selected individual from the population has a value in that interval.

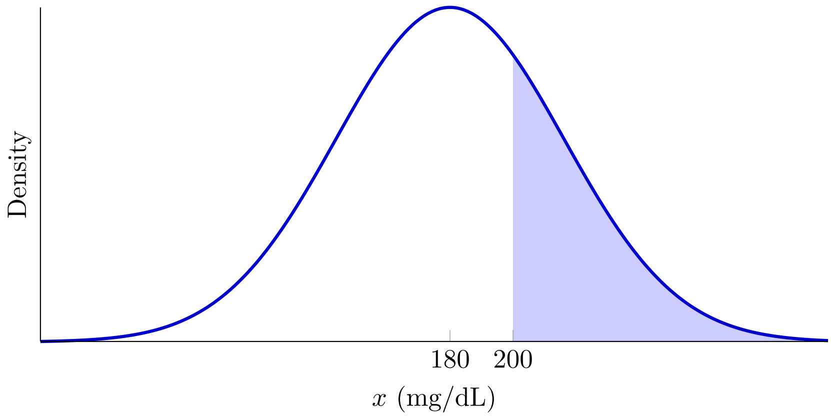
Note

Because the total area under the curve is 1, any area corresponding to an interval of values can be interpreted as a probability.

Example: Interpreting the Area under a Normal Curve

Problem: The serum total cholesterol for males 20–29 years old is approximately normally distributed with mean $\mu = 180$ mg/dL and standard deviation $\sigma = 36.2$ mg/dL, based on data from the National Health and Nutrition Examination Survey.

(a) **Draw a normal curve with parameters labeled.**



(b) **Shade the region to the right of $x = 200$.** Individuals with cholesterol greater than 200 mg/dL are considered to have high cholesterol. On the normal curve, shade the area under the curve for $x > 200$.

(c) **Interpret the area:** Suppose the area under the normal curve to the right of $x = 200$ is 0.2903. This means that approximately 29.03% of males aged 20–29 have total cholesterol greater than 200 mg/dL. Alternatively, the probability that a randomly selected male in this age group has high cholesterol is 0.2903.

End of Lecture #12