MATH 108: Elementary Probability and Statistics

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Probability

The Sample Space and Events

Definition: A *sample space* S of a probability experiment is the set of all possible outcomes.

Definition: An *event* is any collection of outcomes from a probability experiment. An event may consist of one or more outcomes.

Terminology:

- An **outcome** is the result of a single trial of an experiment.
- A **simple event** is an event that consists of exactly one outcome.
- Events are typically denoted with capital letters such as E.

Example: Rolling a Die

Consider a probability experiment where we roll a single fair six-sided die.

- (a) The outcomes of this experiment are:
$$e_1 = 1,\ e_2 = 2,\ e_3 = 3,\ e_4 = 4,\ e_5 = 5,\ e_6 = 6$$
- (b) The sample space is the set of all possible outcomes:
$$S = \{1, 2, 3, 4, 5, 6\}$$
- (c) Let E = "roll an even number". Then:
$$E = \{2, 4, 6\}$$

Rules of Probability

Let $P(E)$ represent the probability that event E occurs.

Probability Rules

1. **Rule 1:** For any event E,
$$0 \le P(E) \le 1$$
2. **Rule 2:** The sum of the probabilities of all outcomes in the sample space must be equal to 1. If $S = \{e_1, e_2, \ldots, e_n\}$, then:
$$P(e_1) + P(e_2) + \cdots + P(e_n) = 1$$

Note:

- Probabilities such as -0.3 or 1.32 are **not possible**, because they violate Rule 1.
- A valid **probability model** must satisfy both Rule 1 and Rule 2.

Example: Probability Model with M&Ms

Suppose a plain M&M candy is randomly selected. The table below shows the probability of each color:

Color	**Probability**
Brown	0.13
Yellow	0.14
Red	0.13
Blue	0.24
Orange	0.20
Green	0.16

Verification:

- Each probability is between 0 and 1 $\Rightarrow$ Rule 1 is satisfied.
- Total probability:
$$0.13 + 0.14 + 0.13 + 0.24 + 0.20 + 0.16 = 1$$
$\Rightarrow$ Rule 2 is satisfied.

Therefore, this is a valid **probability model**.

Understanding Probability Values

- If an event is **impossible**, then $P(E) = 0$.
- If an event is **certain**, then $P(E) = 1$.
- The **closer** a probability is to 1, the more likely the event is to occur.
- The **closer** a probability is to 0, the less likely the event is to occur.

Example: If $P(E_1) = 0.8$ and $P(E_2) = 0.75$, then E_1 is more likely to occur than E_2.

- $P(E_1) = 0.8$ means that, over many repetitions, we expect E_1 to occur about 80 times in 100 trials.
- $P(E_2) = 0.75$ means we expect E_2 to occur about 75 times in 100 trials.

Important: These are long-term expectations. For a small number of trials, the actual frequency may differ.

Law of Large Numbers

The more times a probability experiment is repeated, the closer the relative frequency of an event gets to its theoretical probability.

Unusual Events

Definition: An **unusual event** is an event that has a **low probability** of occurring.

Unusual Events and Contextual Interpretation

Definition: An event is considered **unusual** if it has a low probability of occurring.
However, what qualifies as "low" is often **subjective** and depends on context.

Examples:

- A probability of **5%** ($P = 0.05$) may not be low enough to justify executing a person convicted of a crime, given the high consequence of error (death). We would want the probability of wrongful conviction to be much **closer to 0**.
- A 3% chance of rain ($P = 0.03$) on the day of a picnic would typically be considered an **unusual** event — you'd likely go ahead with the picnic.

Conclusion: There is no strict rule for deciding when an event is unusual. Statisticians often use thresholds like:

$$P < 0.10, \quad P < 0.05, \quad \text{or} \quad P < 0.01$$

Caution: A 5% probability is not always an appropriate cutoff. Use judgment and consider the consequences.

Methods of Determining Probability

There are three main methods used to determine the probability of an event:

1. **Empirical Method (Relative Frequency)**
2. **Classical Method (Theoretical)**
3. **Subjective Method (Personal Judgment)**

We begin with the empirical method.

Empirical Method: Computing Probabilities from Data

Concept: The empirical (or experimental) approach uses observed data to approximate probabilities.
Formula:

$$P(E) \approx \frac{\text{frequency of } E}{\text{number of trials of the experiment}} \tag{1}$$

This method is useful when:

- The theoretical probabilities are unknown or difficult to compute.
- We have data from actual observations or experiments.

Important Note:

The probability calculated using the empirical method is an **estimate**, not an exact value. Results may vary between different sets of trials.

Law of Large Numbers: As the number of trials increases, the relative frequency of an event tends to approach the true (theoretical) probability.

Example: Coin Toss

Suppose you flip a coin 20 times and observe 11 heads.

$$P(\text{Head}) \approx \frac{11}{20} = 0.55$$

Repeat the experiment, and you might get a different value. The more times you repeat the experiment, the more stable the estimate becomes.

Example: Auto Insurance Claims

An insurance agent insures 182 teenage drivers. Last year, 24 of them filed claims.

$$P(\text{Claim}) \approx \frac{24}{182} \approx 0.132$$

This suggests that about 13.2% of teenage drivers are expected to file a claim in a given year. That is, for every 100 teenagers insured, approximately 13 are expected to file a claim.

Why Surveys Are Probability Experiments

Surveys involve randomly selecting individuals and recording responses. Each time a survey is conducted, a different random sample is selected, which may lead to different results.

Therefore: Surveys are considered probability experiments, and their results are subject to variability. Repeating a survey may yield slightly different data due to different respondents.

Computing and Interpreting Probabilities Using the Classical Method

The **empirical method** uses data from actual experiments to approximate probabilities. In contrast, the **classical method** relies on logic and counting — no experiment needs to be performed.

Classical Probability

Definition: The **classical method** applies to experiments with **equally likely outcomes**.

Examples of equally likely outcomes:

- Rolling a fair six-sided die
- Flipping a fair coin
- Drawing a card at random from a well-shuffled standard deck

Caution: If outcomes are **not** equally likely (e.g., a loaded die), the classical method is invalid.

Formula: Classical Probability

If an experiment has n equally likely outcomes, and event E can occur in m of those outcomes, then:

$$P(E) = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in sample space}} = \frac{m}{n} \tag{2}$$

Alternatively, using set notation:

$$P(E) = \frac{N(E)}{N(S)} \tag{3}$$

Where:

- $N(E)$ = number of outcomes in event E
- $N(S)$ = total number of outcomes in the sample space S

Example: Rolling a Fair Die

Let $S = \{1, 2, 3, 4, 5, 6\}$. All outcomes are equally likely.
Let E = "rolling an even number" $= \{2, 4, 6\}$

$$P(E) = \frac{3}{6} = 0.5$$

Building a Probability Model from Survey Data

Problem: 200 individuals were surveyed on their means of travel to work. The results are in Table 2.
Objective:

(a) Build a probability model from the data.

(b) Estimate the probability that a randomly selected person carpools to work.

(c) Determine whether it is unusual for someone to walk to work.

Table 2: Frequency Data

Means of Travel	Frequency
Drive alone	153
Carpool	22
Public transportation	10
Walk	5
Other means	3
Work at home	7

Solution:

(a) Total responses: $153 + 22 + 10 + 5 + 3 + 7 = 200$
Compute relative frequencies (probabilities) for each category:

Means of Travel	Probability
Drive alone	$\frac{153}{200} = 0.765$
Carpool	$\frac{22}{200} = 0.11$
Public transportation	$\frac{10}{200} = 0.05$
Walk	$\frac{5}{200} = 0.025$
Other means	$\frac{3}{200} = 0.015$
Work at home	$\frac{7}{200} = 0.035$

This is the complete **probability model**.

(b) Estimated probability someone carpools:

$$P(\text{Carpool}) = 0.11$$

Interpretation: In a group of 1000 workers, we expect about $0.11 \times 1000 = 110$ to carpool.

(c) Probability someone walks:

$$P(\text{Walk}) = 0.025$$

Since this is close to or below common "unusual" cutoffs (e.g., 0.05), walking to work is considered an **unusual event**.

Conclusion: The classical method is useful when all outcomes are equally likely. When outcomes are not equally likely or when data is available, the empirical method is preferred.

Complementary Event

For every event A, there exists another event A' called the **complementary event** to A. It is also called the event "not A". If S is the sample space, then:

$$A' = \{\omega \in S : \omega \notin A\} = S - A$$

Example

Consider the experiment of tossing three coins. The sample space is:

$$S = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}$$

Let $A = \{\text{HTH, HHT, THH}\}$ represent the event "only one tail appears". Then the complementary event is:

$$A' = \{\text{HHH, HTT, THT, TTH, TTT}\}$$

The Event 'A or B'

The union of two events A and B, denoted by $A \cup B$, represents the event "either A or B or both":

$$A \cup B = \{\omega \in S : \omega \in A \text{ or } \omega \in B\}$$

The Event 'A and B'

The intersection of events A and B, denoted by $A \cap B$, is the event "both A and B occur":

$$A \cap B = \{\omega \in S : \omega \in A \text{ and } \omega \in B\}$$

Example

Experiment: throwing a die twice.

$$\begin{aligned}
A &= \text{first throw is } 6 = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \\
B &= \text{sum is at least } 11 = \{(5,6),(6,5),(6,6)\} \\
A \cap B &= \{(6,5),(6,6)\}
\end{aligned}$$

The Event 'A but not B'

The difference of sets A and B is the event "A occurs but not B":

$$A - B = \{\omega \in A : \omega \notin B\} = A \cap B'$$

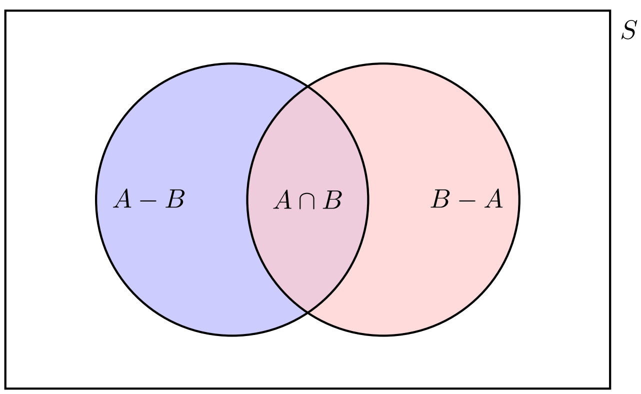
Example

Experiment: rolling a die.

$$\begin{aligned} S &= \{1, 2, 3, 4, 5, 6\} \\ A &= \text{prime number} = \{2, 3, 5\} \\ B &= \text{odd number} = \{1, 3, 5\} \end{aligned}$$

- $A \cup B = \{1, 2, 3, 5\}$
- $A \cap B = \{3, 5\}$
- $A - B = \{2\}$
- $A' = \{1, 4, 6\}$

Venn Diagram



Mutually Exclusive Events

Two events A and B are called **mutually exclusive** if:

$$A \cap B = \varnothing$$

This means that A and B cannot happen at the same time.

Example

Experiment: rolling a die.

$$\begin{aligned} A &= \text{odd number} = \{1,3,5\} \\ B &= \text{even number} = \{2,4,6\} \\ A \cap B &= \varnothing \end{aligned}$$

Thus, A and B are mutually exclusive.

Counter Example

Let $A = \{1,3,5\}$ and $B = \{1,2,3\}$. Then $A \cap B = \{1,3\} \neq \varnothing$; hence, not mutually exclusive.

Exhaustive Events

A set of events $E_1, E_2, ..., E_n$ is called **exhaustive** if their union covers the entire sample space:

$$E_1 \cup E_2 \cup \cdots \cup E_n = S$$

If the events are also mutually exclusive (i.e., $E_i \cap E_j = \varnothing$ for $i \neq j$), then they are called **mutually exclusive and exhaustive** events.

Example

Experiment: rolling a die.

$$\begin{aligned} A &= \text{number} < 4 = \{1,2,3\} \\ B &= \text{number} > 2 \text{ and } < 5 = \{3,4\} \\ C &= \text{number} > 4 = \{5,6\} \end{aligned}$$

$$A \cup B \cup C = \{1,2,3,4,5,6\} = S$$

So, A, B, and C are exhaustive.

Worked Examples

Example 1

Experiment: tossing two dice.

- A: sum is even
- B: sum is a multiple of 3
- C: sum is less than 4
- D: sum is greater than 11

Sample space S has 36 outcomes.

- $C \cap D = \varnothing \Rightarrow C$ and D are mutually exclusive.
- All other intersections are non-empty $\Rightarrow$ not mutually exclusive.

Example 2

Experiment: tossing a coin three times.

$$
\begin{aligned}
&S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}\\
&A = \text{No head} = \{TTT\}\\
&B = \text{Exactly one head} = \{HTT, THT, TTH\}\\
&C = \text{At least two heads} = \{HHT, HTH, THH, HHH\}
\end{aligned}
$$

Then:

- $A \cup B \cup C = S \Rightarrow$ Exhaustive
- All pairwise intersections are $\varnothing \Rightarrow$ Mutually exclusive

Hence, A, B, and C form a set of mutually exclusive and exhaustive events.

End of Lecture #6