MATH 108: Elementary Probability and Statistics

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Probability

Basic Definitions

- **Experiment**: A process that leads to one of several possible outcomes.
- **Event (E)**: A set of outcomes from a probability experiment.

Independent Events

Definition:

Two events A and B are **independent** if the occurrence of one does not affect the probability of the other.

Example 1: Tossing a Coin and Rolling a Die

Let:

- Event A: Flip a coin and get heads.
- Event B: Roll a die and get a 6.

Calculate the sample space

There are:

- 2 outcomes when flipping a coin: {Heads, Tails}
- 6 outcomes when rolling a die: {1, 2, 3, 4, 5, 6}

Since the events are independent, we can form a combined sample space of all possible outcomes:

$$\text{Total outcomes} = 2 \times 6 = 12$$

Favorable outcomes where we get **Heads and 6**:

$$\text{Only one: } (\text{Heads}, 6)$$

So:

$$P(\text{Heads and } 6) = \frac{1}{12}$$

Multiplication Rule

For Independent Events:

$$P(A \cap B) = P(A) \cdot P(B)$$

Example 1: Use the Multiplication Rule

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{6}$$

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

Using the multiplication rule gives the same result with far less work, especially as problems become more complex.

Other Examples of Independent Events:

1. **Two Coin Tosses**
   Let A: First toss is heads, B: Second toss is tails.

$$P(A \cap B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

2. **Roulette Example**
   Probability ball lands on 17 two times in a row:

$$P(\text{17 on both spins}) = \frac{1}{38} \cdot \frac{1}{38} = \frac{1}{1444}$$

3. **Selecting With Replacement**
   Selecting a female student from a class of 10 (4 female, 6 male), replacing, and selecting again:

$$P(\text{female on both}) = \frac{4}{10} \cdot \frac{4}{10} = \frac{16}{100} = 0.16$$

Example 2: Probability of Getting Five Heads in a Row

Problem: What is the probability of obtaining five heads in a row when flipping a fair coin?

Solution

Each flip of a fair coin has two equally likely outcomes: Heads (H) or Tails (T). The probability of getting heads on a single flip is:

$$P(H) = \frac{1}{2}$$

Since each flip is independent, the probability of getting five heads in a row is the product of five independent events:

$$P(\text{5 Heads}) = P(H) \cdot P(H) \cdot P(H) \cdot P(H) \cdot P(H) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$P(\text{5 Heads}) = \boxed{\frac{1}{32}} \approx \boxed{0.03125}$$

Examples of Independent Events and Their Probabilities

Rolling Four Ones in a Row

Problem: What is the probability of obtaining four ones in a row when rolling a fair, six-sided die?
Solution: Each die roll is independent, and the probability of rolling a 1 on one roll is:

$$P(1) = \frac{1}{6}$$

The probability of rolling four ones in a row is:

$$P(4 \text{ ones}) = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

$$P(4 \text{ ones}) = \boxed{\frac{1}{1296}} \approx \boxed{0.0007716}$$

Interpretation: There is less than a 0.1% chance of rolling four 1's in a row. This is a very rare event, and we would not expect it to happen often, even in many repeated trials.

Example 2: Probability of Both People Being Left-Handed

Problem: About 13% of the population is left-handed. If two people are randomly selected:

1. What is the probability that both are left-handed?
2. What is the probability that at least one is right-handed?

Solution:
Let $P(L) = 0.13$ be the probability a person is left-handed. Let $P(R) = 1 - P(L) = 0.87$ be the probability a person is right-handed.

1. Probability both are left-handed:

$$P(\text{both left-handed}) = 0.13 \times 0.13 = \boxed{0.0169}$$

2. Probability that at least one is right-handed:

   Use the complement rule:

$$P(\text{at least one right-handed}) = 1 - P(\text{both left-handed})$$

$$= 1 - 0.0169 = \boxed{0.9831}$$

Interpretation:

- There is only a 1.69% chance that both people are left-handed.
- There is a 98.31% chance that at least one of the two people is right-handed.
- This reflects how uncommon it is to find two left-handed people randomly, given only 13% of the population is left-handed.

Standard 52-Card Deck Summary for Probability Problems

- Total number of cards in a standard deck:
$$N = 52$$
- Number of suits:
$$4 \text{ (Hearts, Diamonds, Clubs, Spades)}$$
- Number of cards per suit:
$$13 \text{ cards per suit: } \{A, 2, 3, \ldots, 10, J, Q, K\}$$
- Number of red cards:
$$26 \text{ (Hearts and Diamonds)}$$
- Number of black cards:
$$26 \text{ (Clubs and Spades)}$$
- Number of face cards:
$$12 \text{ (J, Q, K in each suit: } 3 \times 4\text{)}$$
- Number of Aces:
$$4 \text{ (1 per suit)}$$
- Number of cards with value 10 or higher (10, J, Q, K):
$$16 \text{ cards (4 of each)}$$
- Probability of drawing a card of a specific suit:
$$P(\text{specific suit}) = \frac{13}{52} = \frac{1}{4}$$
- Probability of drawing a face card:
$$P(\text{face card}) = \frac{12}{52} = \frac{3}{13}$$
- Probability of drawing an Ace:
$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$$
- Probability of drawing a red card:
$$P(\text{red card}) = \frac{26}{52} = \frac{1}{2}$$
- Probability of drawing a number card (2 through 10):
$$9 \text{ values} \times 4 \text{ suits} = 36 \Rightarrow P = \frac{36}{52} = \frac{9}{13}$$

With vs. Without Replacement

With Replacement

Definition: After selecting an item, it is returned to the sample space before the next selection.

- The sample space remains the same for each trial.
- Probabilities do not change.
- Events are typically independent.

Example: Drawing a card from a deck, replacing it, then drawing again.

$$P(\text{Ace}) = \frac{4}{52} \text{ each time}$$

Without Replacement

Definition: After selecting an item, it is not returned before the next selection.

- The sample space decreases after each selection.
- Probabilities change.
- Events are typically dependent.

Example: Drawing two cards without replacement:

$$P(\text{1st Ace}) = \frac{4}{52}, \quad P(\text{2nd Ace} \mid \text{1st Ace}) = \frac{3}{51}$$

$$P(\text{Both Aces}) = \frac{4}{52} \cdot \frac{3}{51}$$

Summary Table

Feature	**With Replacement**	**Without Replacement**
Item returned?	Yes	No
Sample space changes?	No	Yes
Events independent?	Usually Yes	Usually No
Probability changes?	No	Yes

Dependent Events

Two events are **dependent** if the outcome of one does affect the probability of the other.
Example: Drawing two cards without replacement.

Disjoint (Mutually Exclusive) Events

Definition: Events A and B are disjoint if they cannot both occur at the same time.

$$P(A \cap B) = 0$$

Important: Disjoint events are always **dependent** because if one occurs, the other cannot.

Comparison Table

Feature	Independent Events	Disjoint Events
Can both occur?	Yes	No
Affect each other?	No	Yes
Formula	$P(A \cap B) = P(A) \cdot P(B)$	$P(A \cap B) = 0$
Are they dependent?	No	Yes

Example 1: Roulette (Repeated Independent Events)

In roulette, the wheel has 38 slots (0, 00, 1–36). What is the probability that the ball lands on 17 two times in a row?

Solution:

Each spin is independent. Let:

$$P(17 \text{ on spin } 1) = \frac{1}{38}, \quad P(17 \text{ on spin } 2) = \frac{1}{38}$$

$$P(17 \text{ on both spins}) = \frac{1}{38} \cdot \frac{1}{38} = \frac{1}{1444} \approx 0.0006925$$

Interpretation: Expected about 7 times in 10,000 trials.

Multiplication Rule for n Independent Events

If events $E_1, E_2, \ldots, E_n$ are independent:

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdot \cdots \cdot P(E_n)$$

Example 2: Survival Probability

Given: Probability a 24-year-old male survives one year = 0.9986.

(a) Probability all 3 survive:

$$P(\text{all 3 survive}) = 0.9986^3 = 0.9958$$

Interpretation: In 1000 samples, 996 will have all survive.

(b) Probability all 20 survive:

$$P(\text{all 20 survive}) = 0.9986^{20} = 0.9724$$

Interpretation: In 1000 samples, 972 will have all survive.

Computing "At Least One" Using the Complement Rule

Problem:

Find the probability that at least one of 1000 males dies in one year.

Solution:

$$P(\text{at least one dies}) = 1 - P(\text{none die}) = 1 - 0.9986^{1000}$$

$$= 1 - 0.2464 = 0.7536$$

Interpretation: In 100 samples, about 75 will have at least one death.

Card Deck Assumptions

- A standard deck has 52 cards.
- 4 suits: Hearts, Diamonds (red), Clubs, Spades (black).
- Each suit has 13 cards: A, 2–10, J, Q, K.
- Face cards: J, Q, K (12 total).
- Aces: 4 total.

With Replacement

Q1. Probability that both cards drawn are Aces (with replacement).

Solution:

$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13}$$
$$P(\text{2 Aces with replacement}) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

Q2. First card is a Heart, second is a King (with replacement).

Solution:

$$P(\text{Heart}) = \frac{13}{52} = \frac{1}{4}, \quad P(\text{King}) = \frac{4}{52} = \frac{1}{13}$$
$$P = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$$

Q3. All three cards drawn are red (with replacement).

Solution:

$$P(\text{red}) = \frac{26}{52} = \frac{1}{2}$$
$$P(\text{3 reds}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Q4. Face card followed by a number card (2–10).

$$P(\text{face}) = \frac{12}{52} = \frac{3}{13}, \quad P(\text{number card}) = \frac{36}{52} = \frac{9}{13}$$
$$P = \frac{3}{13} \times \frac{9}{13} = \frac{27}{169}$$

Without Replacement

Q5. Probability both cards are Kings (without replacement).

$$P(\text{1st King}) = \frac{4}{52}, \quad P(\text{2nd King}) = \frac{3}{51}$$
$$P = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$$

Q6. Queen first, then a Spade (without replacement).

$$P(\text{Queen}) = \frac{4}{52}, \quad P(\text{Spade} \mid \text{Queen not Spade}) = \frac{13}{51}$$
$$P(\text{Spade} \mid \text{Queen is Spade}) = \frac{12}{51}$$

So,

$$P = \frac{1}{13} \times \left(\frac{12}{51} \times \frac{1}{4} + \frac{13}{51} \times \frac{3}{4}\right) = \frac{1}{13} \times \left(\frac{3}{51} + \frac{39}{204}\right) = \text{(mixed case)}$$

Alternatively, using total probability:

$$P = \frac{4}{52} \times \frac{13}{51} = \frac{1}{13} \times \frac{13}{51} = \frac{1}{51}$$

Q7. Probability all 3 cards are red (without replacement).

$$P = \frac{26}{52} \times \frac{25}{51} \times \frac{24}{50} = \frac{26 \cdot 25 \cdot 24}{52 \cdot 51 \cdot 50} = \frac{15600}{132600} = \frac{13}{110}$$

Q8. At least one Ace in two cards (without replacement).

Complement approach:

$$P(\text{no Ace}) = \frac{48}{52} \times \frac{47}{51}$$
$$P(\text{at least one Ace}) = 1 - \frac{48}{52} \cdot \frac{47}{51} = 1 - \frac{2256}{2652} = \frac{396}{2652} = \frac{33}{221}$$

Q9. Face card then number card (without replacement).

$$P = \frac{12}{52} \times \frac{36}{51} = \frac{12 \cdot 36}{2652} = \frac{432}{2652} = \frac{36}{221}$$

Summary

- Use the multiplication rule for independent events: $P(E \cap F) = P(E) \cdot P(F)$
- For multiple independent events: $P(E_1 \cap E_2 \cdots \cap E_n) = \prod_{i=1}^{n} P(E_i)$
- Use the complement rule for "at least one":
  $P(\text{at least one}) = 1 - P(\text{none})$
- Disjoint events: Cannot happen together ($P(A \cap B) = 0$); they are dependent.

End of Lecture #8