MATH 108: Elementary Probability and Statistics

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Conditional Probability

Definition

The **conditional probability** of an event A given that another event B has occurred is denoted by $P(A \mid B)$ and is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided that } P(B) > 0$$

This formula tells us how to update the probability of A in the presence of new information that B has occurred.

Solved Problems

Problem 1: Drawing Cards

A card is drawn at random from a standard 52-card deck. What is the probability that it is a King, given that it is a face card?

Solution:
Face cards include Jacks, Queens, and Kings. Each suit has one of each, so there are:

$$\text{Total face cards} = 3 \times 4 = 12$$

$$\text{Number of Kings} = 4$$

Step 1: Apply the conditional probability formula.

$$P(\text{King} \mid \text{Face}) = \frac{\text{Number of Kings}}{\text{Number of Face Cards}} = \frac{4}{12}$$

$$P(\text{King} \mid \text{Face}) = \frac{1}{3}$$

$$\boxed{P = \frac{1}{3}}$$

Problem 2: Rolling Dice

Two fair six-sided dice are rolled. Define the events:

$$A = \{\text{The sum of the dice is } 9\}, \quad B = \{\text{The first die shows } 4\}$$

We want to find the conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Step 1: Compute $P(B)$.
Since the first die is independent and fair,

$$P(B) = \frac{1}{6}$$

Step 2: Compute $P(A \cap B)$.
For $A \cap B$, the first die must be 4, and the sum must be 9. This happens only if the second die is 5:

$$(4, 5)$$

Total possible outcomes when rolling two dice is $6 \times 6 = 36$, so:

$$P(A \cap B) = \frac{1}{36}$$

Step 3: Calculate the conditional probability.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

$$\boxed{P(A \mid B) = \frac{1}{6}}$$

Problem 3: Colored Balls

A bag contains 3 red, 2 green, and 5 blue balls. One ball is drawn at random.
Define the events:

$$A = \{\text{The ball drawn is red}\}, \quad B = \{\text{The ball drawn is not blue}\}$$

We want to find the conditional probability $P(A \mid B)$.

Step 1: Determine $P(B)$.
Since the ball is not blue, the possible balls are:

$$3 \text{ red} + 2 \text{ green} = 5 \text{ balls}$$

The total number of balls is $3 + 2 + 5 = 10$, so:

$$P(B) = \frac{5}{10} = \frac{1}{2}$$

Step 2: Determine $P(A \cap B)$.

Since event A (red) is a subset of B (not blue):

$$P(A \cap B) = P(A) = \frac{3}{10}$$

Step 3: Calculate the conditional probability.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{10}}{\frac{1}{2}} = \frac{3}{5}$$

$$\boxed{P(A \mid B) = \frac{3}{5}}$$

Problem 4: Exam Passing

In a group of students:

- 60% passed Math: $P(M) = 0.6$
- 70% passed English: $P(E) = 0.7$
- 50% passed both Math and English: $P(M \cap E) = 0.5$

Find the probability that a student passed Math given they passed English, i.e., $P(M \mid E)$.

Using the conditional probability formula:

$$P(M \mid E) = \frac{P(M \cap E)}{P(E)} = \frac{0.5}{0.7} \approx 0.714$$

$$\boxed{P(M \mid E) \approx 0.714}$$

Problem 5: Family Children

A family has two children. Given that at least one of the children is a boy, what is the probability that both children are boys?

Step 1: List all possible equally likely gender combinations:

$$\{BB, BG, GB, GG\}$$

where B = boy, G = girl.

Step 2: Given that at least one child is a boy, exclude the GG outcome:

$$\{BB, BG, GB\}$$

Step 3: Find the probability both children are boys given this condition:

$$P(\text{Both boys} \mid \text{At least one boy}) = \frac{|\{BB\}|}{|\{BB, BG, GB\}|} = \frac{1}{3}$$

$$\boxed{P = \frac{1}{3}}$$

Problem 6: Drawing Without Replacement

A box contains 2 red balls and 3 blue balls. Two balls are drawn sequentially without replacement. What is the probability that the second ball is red given that the first ball drawn is blue?

Step 1: Define events:

$$A = \{\text{Second ball is red}\}, \quad B = \{\text{First ball is blue}\}$$

Step 2: Given the first ball is blue, update the contents:
- Initial balls: 2 red, 3 blue (total 5) - After removing one blue ball (first draw), remaining balls:

$$2 \text{ red}, \quad 2 \text{ blue}, \quad \text{total } 4$$

Step 3: Compute the conditional probability:

$$P(A \mid B) = \frac{\text{Number of red balls left}}{\text{Total balls left}} = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{P(\text{2nd Red} \mid \text{1st Blue}) = \frac{1}{2}}$$

Problem 7: Students and Sports

In a group of students:

- 40% play football: $P(F) = 0.4$
- 50% play basketball: $P(B) = 0.5$
- 25% play both football and basketball: $P(F \cap B) = 0.25$

Find the probability a student plays football given they play basketball:

$$P(F \mid B) = \frac{P(F \cap B)}{P(B)} = \frac{0.25}{0.5} = 0.5$$

$$\boxed{P(F \mid B) = 0.5}$$

Problem 8: Drawing Two Cards

Two cards are drawn sequentially without replacement from a standard 52-card deck.
What is the probability that the second card is a heart given that the first card drawn was a heart?

Step 1: Initial hearts in the deck: 13 hearts out of 52 cards.

Step 2: After drawing one heart, remaining hearts and cards:

$$12 \text{ hearts left}, \quad 51 \text{ cards left}$$

Step 3: Calculate the conditional probability:

$$P(\text{2nd Heart} \mid \text{1st Heart}) = \frac{12}{51}$$

$$\boxed{P = \frac{12}{51}}$$

Problem 9: Tossing Coins

Three fair coins are tossed. Given that at least one head occurs, what is the probability that there are exactly two heads?

Step 1: List all possible outcomes.
There are $2^3 = 8$ total outcomes when tossing 3 coins:

$$\{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}, \text{HTT}, \text{THT}, \text{TTH}, \text{TTT}\}$$

Step 2: Remove the outcome with no heads (TTT).
Since we are told that at least one head occurs, we exclude `TTT`. So, the sample space becomes:

$$S = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}, \text{HTT}, \text{THT}, \text{TTH}\}$$

Total number of outcomes with at least one head: $|S| = 7$

Step 3: Count favorable outcomes (exactly 2 heads).
These are:

$$\text{HHT}, \ \text{HTH}, \ \text{THH}$$

So, there are 3 favorable outcomes.

Step 4: Compute the conditional probability.

$$P(\text{Exactly 2 Heads} \mid \text{At least 1 Head}) = \frac{3}{7}$$

Problem 10: Survey on TV Preference

In a survey:

- 70% of people like Channel A: $P(A) = 0.7$
- 60% like Channel B: $P(B) = 0.6$
- 50% like both channels: $P(A \cap B) = 0.5$

What is the probability that a person likes Channel A given that they like Channel B?

Step 1: Use the formula for conditional probability.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Step 2: Plug in the known values.

$$P(A \mid B) = \frac{0.5}{0.6} = \frac{5}{6}$$

$$\boxed{P = \frac{5}{6}}$$

Problem 11: Die Roll Divisibility

A standard six-sided die is rolled. Given that the result is even, what is the probability that it is divisible by 3?

Step 1: Identify the sample space.
The full set of outcomes when rolling a die is:

$$\{1, 2, 3, 4, 5, 6\}$$

Step 2: Restrict to even outcomes.
Given that the result is even, the possible outcomes are:

$$E = \{2, 4, 6\}$$

So, there are $|E| = 3$ possible outcomes under this condition.

Step 3: Find favorable outcomes (even and divisible by 3).
Only one of the even numbers is divisible by 3:

$$\{6\}$$

So, there is 1 favorable outcome.

Step 4: Compute the conditional probability.

$$P(\text{Divisible by 3} \mid \text{Even}) = \frac{1}{3}$$

$$\boxed{P = \frac{1}{3}}$$

Problem 12: Bag with Two Colors

A bag contains 4 red balls and 6 blue balls. One ball is drawn at random without replacement, and then a second ball is drawn. What is the probability that the second ball is red, given that the first ball was blue?

Step 1: Understand the total and initial setup.
Total balls: $4 + 6 = 10$

$$\text{Initial contents: } \{4 \text{ red},\ 6 \text{ blue}\}$$

Step 2: Condition — the first ball drawn is blue.
If the first ball is blue, we are left with:

$$\text{Remaining balls: } 4 \text{ red, } 5 \text{ blue} \Rightarrow 9 \text{ total}$$

Step 3: Probability second is red given first was blue.
There are 4 red balls among 9 remaining:

$$P(\text{Second is red} \mid \text{First is blue}) = \frac{4}{9}$$

$$\boxed{P = \frac{4}{9}}$$

Problem 12: Dice and Primes

A standard six-sided die is rolled. Given that the result is a prime number, what is the probability that it is less than 4?

Step 1: Identify the prime numbers on a die.
The prime numbers from 1 to 6 are:

$$\text{Prime outcomes} = \{2, 3, 5\}$$

So, there are 3 outcomes in the conditional sample space.

Step 2: Find favorable outcomes (less than 4).
Among the prime numbers, those less than 4 are:

$$\{2, 3\}$$

That gives us 2 favorable outcomes.

Step 3: Compute the conditional probability.

$$P(\text{Result} < 4 \mid \text{Prime}) = \frac{2}{3}$$

$$\boxed{P = \frac{2}{3}}$$

Section 5.5: Counting Techniques

Objectives

1. Solve counting problems using the **Multiplication Rule**.
2. Solve counting problems using **permutations**.
3. Solve counting problems using **combinations**.
4. Solve counting problems involving **permutations with nondistinct items**.
5. Compute **probabilities involving permutations and combinations**.

Key Concepts Overview

- **Multiplication Rule:** If an event consists of a sequence of stages, and each stage can occur in a certain number of ways, the total number of outcomes is the product of the number of choices at each stage.
- **Permutations:** Used when selecting and arranging objects where the order *matters*.
- **Combinations:** Used when selecting objects where the order *does not matter*.
- **Permutations with Nondistinct Items:** Adjusts the standard permutation formula to account for identical objects.
- **Probability Applications:** These techniques can be used to compute probabilities by determining the number of favorable outcomes over the number of possible outcomes.

Problem:

Find the number of 4-letter words, with or without meaning, which can be formed from the letters of the word **ROSE**, where repetition of letters is not allowed.

Solution:

The word "ROSE" contains 4 distinct letters: R, O, S, E.
We are to form 4-letter words using all of these letters without repetition, and order matters (since "ROSE" and "SORE" are considered different).
This is a permutation of 4 distinct letters taken all at once:

$$\text{Number of 4-letter words} = P(4,4) = 4! = 24$$

Final Answer:

$$\boxed{24}$$

Note:

If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words $= 4 \times 4 \times 4 \times 4 = 256$.

1. Permutations

Definition: A **permutation** is an arrangement of items in a specific order. Use permutations when the **order matters**.
Formula:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Where:

- n = total number of items
- r = number of items chosen and arranged

Examples:

1. How many ways can 3 books be arranged on a shelf?

$$P(3,3) = 3! = 6$$

2. How many different 2-digit codes can be made using the digits 1, 2, 3 (no repetition)?

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$$

3. A password is made by choosing 3 different letters from A, B, C, D. How many possible passwords (in order)?

$$P(4,3) = \frac{4!}{(4-3)!} = \frac{24}{1} = 24$$

2. Combinations

Definition: A **combination** is a selection of items where **order does not matter**. Use combinations when the arrangement doesn't matter.
Formula:

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Where:

- n = total number of items
- r = number of items chosen

Examples:

1. How many ways can you choose 2 fruits from apple, banana, and cherry?

$$C(3,2) = \binom{3}{2} = \frac{3!}{2!1!} = 3$$

2. From a group of 5 students, how many ways can you choose 3 to form a team?

$$C(5,3) = \binom{5}{3} = \frac{5!}{3!2!} = 10$$

3. A committee of 2 people is to be formed from 4 candidates. How many different committees are possible?

$$C(4,2) = \binom{4}{2} = \frac{4!}{2!2!} = 6$$

3. Permutations with Identical Items

Definition: When some items are identical, the total number of unique permutations must account for repeated elements.

Formula:

$$\text{Permutations} = \frac{n!}{k_1! \cdot k_2! \cdot \cdots \cdot k_r!}$$

Where:

- n is the total number of items,
- $k_1, k_2, \ldots, k_r$ are the frequencies of each identical item.

Example 1:

1. How many different ways can the letters in the word "MOM" be arranged?

$$\text{Total letters: } 3, \quad \text{M appears twice}$$
$$\text{Permutations} = \frac{3!}{2!} = \frac{6}{2} = 3$$

Example 2:

For the word `BALLOON`, which has 7 letters with:

- L appearing twice,
- O appearing twice,

$$\text{Permutations} = \frac{7!}{2! \cdot 2!} = \frac{5040}{4} = 1260$$

Problem:

How many 4-digit numbers can be formed using the digits 1 to 9, if repetition of digits is not allowed?

Solution:

There are 9 digits available: 1, 2, 3, 4, 5, 6, 7, 8, 9 (no zero). Since repetition is not allowed, we are selecting and arranging 4 different digits.

This is a permutation of 4 digits chosen from 9:

$$\text{Number of 4-digit numbers} = P(9,4) = \frac{9!}{(9-4)!} = \frac{9!}{5!}$$

$$= \frac{362880}{120} = 3024$$

Final Answer:

$$\boxed{3024}$$

Problem: Guiseppe's Pizza Deal

Guiseppe's Pizza deal includes a large pizza with unlimited toppings for \$10. There are 8 toppings to choose from: sausage, pepperoni, mushroom, olive, tomato, spinach, onion, and extra cheese.

(a) Probability a 2-topping pizza includes pepperoni

We are choosing 2 toppings from 8. All combinations are equally likely.
Total possible 2-topping pizzas:

$$\binom{8}{2} = 28$$

Favorable outcomes (including pepperoni):
Fix pepperoni as one topping, then choose 1 topping from the remaining 7:

$$\binom{7}{1} = 7$$

Probability:

$$P(\text{pepperoni}) = \frac{7}{28} = \frac{1}{4}$$

(b) Probability at least one of your two pizzas is free

Out of 100 pizzas, 5 are randomly selected to be free. You order 2 pizzas. What is the probability that **at least one** is free?
This is a hypergeometric probability problem.
Total pizzas: $N = 100$
Free pizzas (successes): $K = 5$
Pizzas you order (draws): $n = 2$
Step 1: Compute $P(\text{no free pizzas})$

$$P(\text{no free}) = \frac{95}{100} \cdot \frac{94}{99} = \frac{8930}{9900}$$

Step 2: Final Answer

$$P(\text{at least one free}) = 1 - \frac{8930}{9900} = \frac{970}{9900} = \frac{97}{990} \approx 0.098$$

Answer: $\boxed{\frac{97}{990}}$

End of Lecture #9