CALCULUS I

ATUL ANURAG
EMAIL: AA2894@NJIT.EDU
WEBSITE: ATULANURAG.COM

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1. Math 111 EXAM I, Spring 2022

1.1. **Problem 1.** Evaluate the following limits:

$$a.\ \lim_{x\to 0} \frac{\tan 3x^2}{x\sin(x)} \qquad\qquad b.\ \lim_{x\to+\infty} \frac{\sqrt{4x^2+2x}+4x}{2x-1}$$

1.2. **Problem 2.** Evaluate the following limits:

$$a.\lim_{x\to 0}\left(\frac{\sin(4x)}{2x}-\frac{2x}{\sin(4x)}\right) \qquad\qquad b.\ \lim_{x\to 1^-}\frac{2+x}{x^2-3x+2}$$

1.3. **Problem 3.** Evaluate the following limits:

$$a.\ \lim_{t\to 3}\frac{t^2+2t-15}{t^2-2t-3} \qquad\qquad b.\ \lim_{x\to 2}\frac{x-2}{\sqrt{x+7}-3}.$$

1.4. **Problem 4.** (a) Find the average rate of change of $f(x)=\sqrt{9+5x^3}$ over the interval $-1\le x\le 2$.

(b) Use the definition of the derivative to find the derivative of $f(x)=\frac{1}{x}$.

1.5. **Problem 5.** Find the constants a and b so that the function given below is continuous for all x:

$$f(x)=\begin{cases} a\,\frac{\sin(2x)}{x},\ x<0\\ x+b,\ 0\le x\le 1\\ \frac{x^2-1}{x-1},\ x>1\end{cases}$$

1.6. **Problem 6.** Find an equation for all asymptotes, if they exist, for the following function, and be sure to label the type of asymptote (horizontal, vertical or oblique (slant)):

$$y=\frac{6x^2+8x-5}{2x+4}.$$

1.7. **Problem 7.** Find all points where the following function is discontinuous and identify the type of discontinuity (jump, infinite or removable). If the function has a removable discontinuity, then determine how to define f at that point in a way that extends $f(x)$ to be continuous there,

$$y=\frac{x^2+2x}{x^2+6x+8}.$$

2. Math 111 Exam I Solutions, Spring 2022

2.1. **Solution 1 (a) and (b).** We know that $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$, and $\lim_{\theta\to 0}\frac{\tan(\theta)}{\theta}=1$.

a.

$$\lim_{x\to 0}\frac{\tan 3x^2}{x\sin(x)}\left(\frac{0}{0}\right)\ \mathit{form}\ =3\lim_{x\to 0}\frac{\frac{\tan 3x^2}{3x^2}}{\frac{\sin(x)}{x}}=3.$$

b.

$$\lim_{x\to+\infty}\frac{\sqrt{4x^2+2x}+4x}{2x-1}=\lim_{x\to+\infty}\frac{\sqrt{4+2/x}+4}{2-1/x}=3.$$

2.2. **Solution 2(a) and (b).** We know that $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$.

a.

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sin(4x)}{2x} - \frac{2x}{\sin(4x)}\right) &= \lim_{x\to 0}\frac{\sin(4x)}{2x} - \lim_{x\to 0}\frac{2x}{\sin(4x)}\\ &= 2\lim_{x\to 0}\frac{\sin(4x)}{4x} - \lim_{x\to 0}\frac{1}{2\frac{\sin(4x)}{4x}}\\ &= 2 - 1/2\\ &= 3/2.\end{aligned}$$

b.

$$\lim_{x\to 1^-}\frac{2+x}{x^2-3x+2} = \lim_{x\to 1^-}\frac{2+x}{(x-1)(x-2)} = +\infty.$$

Approaching from the left side to the point $x = 1$ means putting random values close to 1, but less than 1(and checking the sign of the function if it is positive or negative).

- If the sign is negative, the function approaches to $-\infty$.
- If the sign is positive, the function approaches to $+\infty$.

From the graph below, it is clear that the function $f(x) \to +\infty$, as $x \to 1^-$.

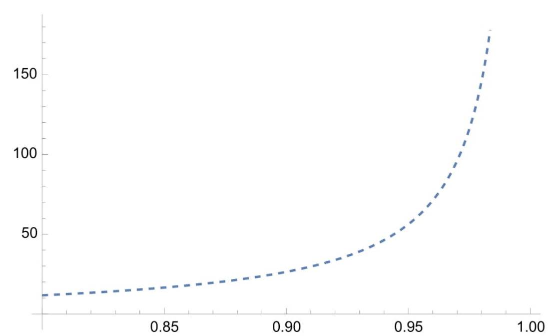

FIGURE 1. $f(x) = \frac{x+2}{x^2-3x+2}$

2.3. **Solution 3(a) and (b).**

a.

$$\lim_{t\to 3}\frac{t^2+2t-15}{t^2-2t-3} = \lim_{t\to 3}\frac{(t+5)(t-3)}{(t-3)(t+1)} = \lim_{t\to 3}\frac{t+5}{t+1} = \frac{8}{4} = 2.$$

b.

$$\begin{aligned}\lim_{x\to 2}\frac{x-2}{\sqrt{x+7}-3} &= \lim_{x\to 2}\frac{(x-2)(\sqrt{x+7}+3)}{(\sqrt{x+7}-3)(\sqrt{x+7}+3)}\\ &= \lim_{x\to 2}\frac{(x-2)(\sqrt{x+7}+3)}{x-2}\\ &= 6.\end{aligned} \tag{1}$$

2.4. **Solution 4(a) and (b).** a. The average rate change of the function $f(x)$ over the interval $a \le x \le b$ is given by

$$\frac{f(b)-f(a)}{b-a} = \frac{\sqrt{9+5(2)^3}-\sqrt{9+5(-1)^3}}{2-(-1)} = \frac{7-4}{3} = 1.$$

b. Using the definition of the derivative we have,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{-h}{x(x+h)h} = -\frac{1}{x^2}.$$

2.5. **Solution 5.** By definition of continuity, a function, $f(x)$ is continuous at $x = a$ if

$$\boxed{\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)}$$

In order to check the continuity for the above function, we need to check its continuity at $x = 0$ and $x = 1$.
Continuity at $x = 0$.

$$\lim_{x\to 0^-} a\frac{\sin 2x}{x} = \lim_{x\to 0^+} x+b = f(0) \tag{2}$$

$$\implies 2a = b \tag{3}$$

Continuity at $x = 1$.

$$\lim_{x\to 1^-} x+b = \lim_{x\to 1^+}\frac{x^2-1}{x-1} = f(1) \tag{4}$$

$$\implies b+1 = 2 \tag{5}$$

Solving (3) and (5), we get,

$$\boxed{a = 1/2,\ b = 1}.$$

2.6. The line $x = a$ is a vertical asymptote of the function

$$y = \frac{6x^2+8x-5}{2x+4},$$

if the limit of the function (one-sided) at this point is infinite.

In other words, it means that possible points are points where the denominator equals 0 or doesn't exist. So, find the points where the denominator equals 0, which is $x = -2$, and

$$\lim_{x\to -2^+}\frac{6x^2+8x-5}{2x+4} = \infty.$$

Since the limit is infinite, then $x = -2$ is a vertical asymptote.

Line $y = b$ is a horizontal asymptote of the function if

$$\lim_{x\to +\infty}\frac{6x^2+8x-5}{2x+4} = \infty \text{ or } \lim_{x\to -\infty}\frac{6x^2+8x-5}{2x+4} = -\infty,$$

and b is finite. Thus, there are no horizontal asymptotes. Slant Asymptotes:

$$y = \frac{6x^2+8x-5}{2x+4} = 3x-2+\frac{3}{2x+4} \tag{6}$$

In equation (6), the rational term approaches 0 as the variable approaches infinity. Thus, the slant asymptote is $y = 3x - 2$.

- Vertical Asymptotes, $x = -2$.
- No horizontal asymptotes.
- Slant Asymptotes, $y = 3x - 2$.

Types of discontinuity:

- A function $f(x)$ has a removable discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists and finite, but the function is either undefined at $x = a$ or $\lim_{x\to a} f(x) \neq f(a)$.
- $f(x)$ has a jump discontinuity if one-sided limits exist and finite but not equal.
- $f(x)$ has a essential discontinuity if one or both of the one-sided limits do not exist or are infinite.

We have

$$f(x) = \frac{x^2+2x}{x^2+6x+8} = \frac{x(x+2)}{(x+2)(x+4)} \tag{7}$$

The function in eq (7) is not well defined at $x = -2$, and $x = -4$.
Continuity at $x = -2$.

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2+2x}{x^2+6x+8}\left(\frac{0}{0}\right) = \lim_{x\to -2} \frac{2x+2}{2x+6} = -1.$$

Since the limit, $\lim_{x\to -2} f(x)$ exists, and is finite. The function can be re-defined at the point $x = -2$. Therefore, $x = -2$ is a removable discontinuity.
Redefining the function, $f(x)$, at $x = -2$ in order to make it continuous.

$$f(x) = \begin{cases} \frac{x^2+2x}{x^2+6x+8}, & x \neq -2 \\ -1, & x = -2 \end{cases}$$

Continuity at $x = -4$.

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} \frac{x^2+2x}{x^2+6x+8} = \lim_{x\to -4} \frac{x}{x+4} = -\infty.$$

Since the limit, $\lim_{x\to -4} f(x)$ exists, and is infinite. Therefore, $x = -4$ is a essential discontinuity.

3. Math 111 EXAM II, Spring, 2022

3.1. Problem 1. Find $\frac{dy}{dx}$ for the following: (a) $y = 1 + \sqrt{x} + x^3$ (b) $y = \sec(x^2)\tan(x^2)$

3.2. Problem 2. (a) Find the second derivative of $y = \ln\left(\frac{2}{e^{x^2}}\right)$.
(b) Let $f(x) = x^3 + x^5, x > 0$. Find the value of $\frac{d}{dx} f^{-1}$ at $x = 2 = f(1)$.

3.3. **Problem 3.** (a) Find the slope of the curve $x^2y = y^3 + 3$ at the point $(2, 1)$.
(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$y = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

3.4. **Problem 4.** (a) At time $t \geq 0$, the position of a body moving along the s-axis is $s = 3 - 2t + e^t$. Determine the time when the body changes direction. What are the body's position and acceleration at this time?
(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and simplify your expression:

$$y = \arcsin(x) + \arcsin\left(\sqrt{1 - x^2}\right), \quad (x > 0).$$

3.5. **Problem 5.** (a) Find $y'(x)$:

$$y = \arctan\left(2^x\right).$$

(b) Find the equation of the tangent line to the curve $x^y = e^x$ at $x = e$.

3.6. **Problem 6.** (a) Suppose $u(x)$ is differentiable at $x = 3$ and $u(3) = 9, u'(3) = -4$. Find the following derivative at $x = 3$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\sqrt{u}\right)$$

(b) Find all points (x, y) on the graph of $y = x/(x - 4)$ with tangent lines perpendicular to the line $y = 4x + 3$.

3.7. **Problem 7.** The area of an expanding circle is increasing at a rate of 12 in^2/s. How fast is the radius increasing when the circumference is 4 in?

4. Math 111 Exam II Solutions, Spring 2022

4.1. **Solution to 1(a) and (b).** a.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(1 + \sqrt{x} + x^3)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(1) + \frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x}) + \frac{\mathrm{d}}{\mathrm{d}x}(x^3) = \frac{1}{2\sqrt{x}} + 3x^2.$$

b.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\sec\left(x^2\right)\tan\left(x^2\right)\Big) &= \sec(x^2)\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x^2)) + \tan(x^2)\frac{\mathrm{d}}{\mathrm{d}x}(\sec(x^2))\\ &= 2x\Big(\sec(x^2)\Big)^2 + 2x\sec(x^2)\Big(\tan(x^2)\Big)^2\\ &= 2x\left[\Big(\sec(x^2)\Big)^2 + \sec(x^2)\Big(\tan(x^2)\Big)^2\right].\end{aligned}$$

4.2. **Solution to 2(a) and (b).** We know that,

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b), \quad \ln\left(e^{f(x)}\right) = f(x).$$

a.

$$y = \ln\left(\frac{2}{e^{x^2}}\right) = \ln(2) - \ln\left(e^{x^2}\right) = \ln(2) - x^2,$$

$$y' = -2x, \quad y'' = -2.$$

b. We know that,

$$f^{-1}(f(x)) = x \tag{8}$$

Differentiating eq (8) with respect to x(using the chain rule), we get,

$$\frac{\mathrm{d}f^{-1}}{\mathrm{d}x} = \frac{1}{f'(x)} \tag{9}$$

We have, $f(x) = x^3 + x^5$. Using eq (9), we get

$$\left.\frac{\mathrm{d}f^{-1}}{\mathrm{d}x}\right|_{x=2} = \left.\frac{1}{f'(x)}\right|_{x=2} = \left.\frac{1}{3x^2+5x^4}\right|_{x=2} = \frac{1}{92}.$$

4.3. **Solution to 3(a) and (b).** a. Slope of the curve $x^2y = y^3 + 3$ at the point $(2, 1)$, is

$$\implies \frac{\mathrm{d}}{\mathrm{d}x}(x^2y) = \frac{\mathrm{d}}{\mathrm{d}x}(y^3+3)$$

$$\implies x^2y' + 2xy = 3y^2y'$$

Solving for y', we get,

$$y' = \frac{2xy}{3y^2 - x^2} \implies y'\Big|_{x=2,y=1} = -4.$$

b. We know that,

$$1 - \cos(2x) = 2\sin^2(x), \qquad 1 + \cos(2x) = 2\cos^2(x)$$

Therefore,

$$y = \frac{1-\cos(2x)}{1+\cos(2x)} = \frac{2\sin^2(x)}{2\cos^2(x)} = \tan^2(x)$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{d}{\mathrm{d}x}\tan^2(x) = 2\tan(x)\sec^2(x)}.$$

4.4. **Solution to 4(a) and (b).** a.

b. Observe,

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\sqrt{1-x^2}\right).$$

We have given,

$$y = \arcsin(x) + \arcsin\left(\sqrt{1-x^2}\right)$$

Using the above observation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

4.5. **Solution to 5(a) and (b).** a. Note,

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln(a),\ \ a > 0.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(2^x)) = \frac{1}{1+(2^x)^2}\frac{d}{\mathrm{d}x}(2^x) = \frac{2^x\ln(2)}{1+(2^x)^2}.$$

b. Taking natural logarithm to both sides of $x^y = e^x$, we get $y\ln(x) = x$. At $x = e, y = e$. Differentiating both sides we get,

$$\frac{y}{x} + \ln(x)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \implies \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(e,e)} = 0.$$

4.6. **Solution to 6(a) and (b).** a. Given, $u(3) = 9,\ u'(3) = -4$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\sqrt{u}) = \frac{x^2u'}{2\sqrt{u}} + 2x\sqrt{u}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}(x^2\sqrt{u})\right|_{x=3} = \frac{3^2u'(3)}{2\sqrt{u(3)}} + 6\sqrt{u(3)} = 12.$$

b. The slope of a line perpendicular to $y = mx + b$ is equal to $-\frac{1}{m}$, where slope m of the equation $y = 4x + 3$ is $m = 4$.

So solve for the points at which the first derivative is equal to $-\dfrac{1}{m} = -\dfrac{1}{4}$. You'll get a quadratic equation, from which you can find the two solutions.

$$y' = \frac{-4}{(x-4)^2} = -\frac{1}{4}$$

$$(x-4)^2 = 16$$

$$x - 4 = \pm 4 \implies x = 0, x = 8$$

4.7. **Solution 7.** We have the area of the circle as

$$A = \pi r^2.$$

Treating A and r as implicitly differentiable functions of t, we get

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}.$$

We are given that

$$c = 2\pi r = 4 \Rightarrow r = \frac{1}{2\pi}.$$

We are also given that,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 12.$$

Putting this together,

$$\implies 12 = 2\pi\frac{1}{2\pi}\frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow 1 = 2\frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = 12.$$

5. Math 111 Exam III, Spring 2022

5.1. Problem 1. Evaluate the following limits:

$$\text{a. } \lim_{x\to 1} \frac{\ln x}{x^4 - x^2}, \quad \text{b. } \lim_{x\to 0} (\cos 2x - \sin 2x)^{1/x}$$

5.2. Problem 2. a. Find the linearization of $f(x) = \ln(1 + 4x + 2x^2)$ about $a = 0$.
b. Use finite approximation to estimate the area under the graph of $f(x) = 4x^2$ between $x = 0$ and $x = 1$ using a lower sum with two rectangles of equal width.

5.3. Problem 3. Find the absolute maximum and minimum values of the following function on the given interval:

$$f(x) = x^{3/2} - 3\sqrt{x}, \quad 0 \le x \le 4.$$

5.4. Problem 4. A rectangular plot of land will be bounded on one side by a stream and the other three sides by a fence. With 40ft of fence at your disposal, what is the largest area you can enclose? Show that your result is a maximum.

5.5. Problem 5. a. Use Newton's method to estimate a solution of $f(x) = x^4 - 2x + 2 = 0$. Start with $x_0 = 0$ and then find x_2.
b. Find the most general antiderivative or indefinite integral:

$$\int (1 + \sec\theta)\cos\theta d\theta$$

5.6. Problem 6. Evaluate the following limits:

$$\text{a. } \lim_{x\to 0} \frac{e^{x^2} - 1}{\cos x - 1}, \qquad \text{b. } \lim_{x\to\infty} \left(\sqrt{9x^2 + 12x} - 3x\right)$$

5.7. Problem 7. Consider the function $y = \frac{x^2+1}{x}$.
a. Find the intervals on which this function is increasing or decreasing
b. Find the intervals on which this function is concave up or concave down
c. Find all asymptotes
d. Determine the points (if any) at which this function has a local maximum, a local minimum or a point of inflection
e. Sketch this function making sure to label the points found in part **d.**

6. Math 111 Exam III Solutions, Spring 2022

6.1. Solution to 1(a, b). a.

$$\lim_{x\to 1} \frac{\ln x}{x^4 - x^2} \left(\frac{0}{0} \text{ form}\right) = \lim_{x\to 0} \frac{1}{x(4x^3 - 2x)} = \frac{1}{2}.$$

b. Let $y = (\cos 2x - \sin 2x)^{1/x}$, taking natural logarithm both sides, we get,

$$\ln(y) = \frac{\ln(\cos 2x - \sin 2x)}{x} \tag{10}$$

Taking $\lim_{x\to 0}$ both sides in equation (10), we get,

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{\ln(\cos 2x - \sin 2x)}{x} \left(\frac{0}{0} \text{ form}\right) = \lim_{x\to 0} \frac{-2(\sin 2x + \cos 2x)}{\cos 2x - \sin 2x} = -2 \tag{11}$$

Taking exponential both sides in equation (11), we get,

$$\lim_{x\to 0} y = \lim_{x\to 0}(\cos 2x - \sin 2x)^{1/x} = e^{-2}.$$

6.2. **Solution to 2(a, b).** a. The linearization of a function $f(x)$ about a point x=a is given by

$$f(x) = f(a) + (x-a)f^{'}(a). \tag{12}$$

For the given function, we have,

$$f(0) = \ln(1) = 0, \quad f^{'}(0) = \left.\frac{4+4x}{1+4x+2x^2}\right|_{x=0} = 4$$

Using (12), the linearization of $f(x) = \ln\left(1+4x+2x^2\right),$ about $x = a = 0$ is,

$$\begin{aligned} f(x) &= f(0) + (x-0)f^{'}(0) \\ &= 0 + 4x \\ &= 4x. \end{aligned}$$

b. We divide the interval $[0,1]$ into the two intervals $[0,1/2]$ and $[1/2,1]$. Notice that this means the width of the rectangles, Δx, is 1/2. Since this is a lower sum, for the interval $[0,1/2]$, we will be approximating f by the value $f(0) = 0$, and for the interval $[1/2,1]$, we will be approximating f by the value $f(1/2) = 1$. Our estimate of the area is then:

$$A \approx f(0)\Delta x + f(1/2)\Delta x = 1/2.$$

6.3. **Problem 3.** The critical points of the function, $f(x) = x^{3/2} - 3\sqrt{x}$ are the roots or where the function $f^{'}(x)$ is undefined.
Here,

$$f^{'}(x) = \frac{3}{2}\sqrt{x} - \frac{3}{2\sqrt{x}} = \frac{3}{2}\,\frac{x-1}{\sqrt{x}}$$

Set $f'(x) = 0$, we get, $x = 0, 1$ as the critical points. Now we evaluate the function at the critical points and the end points of the interval. Absolute extrema are the largest and smallest the function will ever be and these three points represent the only places in the interval where the absolute extrema can occur.

$$f(0) = 0, \quad f(1) = -2, \quad f(4) = 2.$$

Therefore, absolute minimum occurs at $x = 1$, and absolute maximum is at $x = 4$.

6.4. **Solution to 5(a, b).** a. The approximation of a function, $f(x)$, with a initial guess x_0 using the Newton's method is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{13}$$

We have, $x_0 = 0, f(x) = x^4 - 2x + 2 \implies f^{'}(x) = 4x^3 - 2$ Using (13),

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = 1,$$

Now, using $x_1 = 1$, and $n = 1$ in (13), we get,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1}{2} = \frac{1}{2}.$$

b.

$$\int (1 + \sec\theta)\cos\theta d\theta = \int (\cos\theta + 1)\mathrm{d}\theta = \sin\theta + \theta.$$

6.5. **Solution to 6(a, b).** a.

$$\lim_{x\to 0} \frac{e^{x^2} - 1}{\cos x - 1}\left(\frac{0}{0} \text{ form}\right) = -2\lim_{x\to 0} \frac{2xe^{x^2}}{\sin x} = -2\lim_{x\to 0} \frac{e^{x^2}}{\frac{\sin(x)}{x}} = -2.$$

b.

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{9x^2+12x} - 3x\right) &= \lim_{x\to\infty}\left(\sqrt{9x^2+12x} - 3x\right)\frac{\left(\sqrt{9x^2+12x}+3x\right)}{\left(\sqrt{9x^2+12x}+3x\right)} \\
&= \lim_{x\to\infty}\frac{12x}{\left(\sqrt{9x^2+12x}+3x\right)} \\
&= \lim_{x\to\infty}\frac{12}{\left(\sqrt{9+\frac{12}{x}}+3\right)} \\
&= \frac{12}{\sqrt{9}+3} \\
&= 2.
\end{aligned}$$

6.6. **Solution to 7(a, b, c, d, e).** To check where the function y is increasing or decreasing depends on the sign of the derivative y.

a.

$$y' = \frac{x^2 - 1}{x^2}, \tag{14}$$

the critical points are $x = 0, \pm 1$.

- $y' > 0$ on $(-\infty, -1)$ and $(1, +\infty)$. Therefore, the function is increasing on the interval $(-\infty, -1) \cup (1, +\infty)$.
- $y' < 0$ on $(-1, 0)$ and $(0, 1)$. Therefore, the function is increasing on the interval $(-1, 0) \cup (0, 1)$.

b. The function y is concave down in the interval where $y'' < 0$ and concave up where $y'' > 0$. We have,

$$y'' = \frac{2}{x^3}$$

- $y'' < 0$ on $(-\infty, 0)$: concave down,
- $y'' > 0$ on $(0, \infty)$: concave up.

c. The line $x = a$ is a vertical asymptote of the function

$$y = \frac{x^2 + 1}{x},$$

if the limit of the function (one-sided) at this point is infinite.

In other words, it means that possible points are points where the denominator equals 0 or doesn't exist.
So, find the points where the denominator equals 0, which is $x = 0$, and

$$\lim_{x\to 0^+} \frac{x^2+1}{x} = \infty.$$

Since the limit is infinite, then $x = 0$ is a vertical asymptote.
Line $y = b$ is a horizontal asymptote of the function if

$$\lim_{x\to +\infty} \frac{x^2+1}{x} = \infty \text{ or } \lim_{x\to -\infty} \frac{x^2+1}{x} = -\infty,$$

and b is finite.
Thus, there are no horizontal asymptotes.
Slant Asymptotes:

$$y = \frac{x^2+1}{x} = x + \frac{1}{x} \tag{15}$$

In equation (15), the rational term approaches 0 as the variable approaches infinity.
Thus, the slant asymptote is $y = x$.

- Vertical Asymptotes, $x = 0$.
- No horizontal asymptotes.
- Slant Asymptotes, $y = x$.

d. The critical point(s) of y where

$$y'' > 0$$

is the local minima and the critical point(s) y where

$$y'' < 0$$

is the local maxima. Here, the critical points of y are $x = 0, \pm 1$ and $y'' = \frac{2}{x^3}$. Therefore, y has a local minima at $x = -1$ and local maxima at $x = 1$.

To find the point of inflection, we need to find the values of x where

$$y'' = 0.$$

We have,

$$y'' = \frac{2}{x^3} = 0,$$

there are no values of x that can satisfy the above equation. There, y has no point(s) of inflection.

7. Final Exam-Spring 2022

7.1. **Problem 1.** Evaluate the following limits:
(a) $\lim_{x\to 0} \frac{\sin x^2}{\cos 2x-1}$ (b) $\lim_{x\to 0} \left(e^{2x}+x\right)^{\frac{1}{x}}$ (c) $\lim_{t\to -2} \frac{t^3-2t+4}{t^2-t-6}$

7.2. **Problem 2.** Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following.

$$(a)\ y = e^{x+y} - x \quad (b)\ y = x\arcsin(2x) \quad (c)\ y = x^{1+\frac{1}{\ln x}}$$

7.3. **Problem 3.** Evaluate the integrals:

$$(a)\int_{-1}^{1}\frac{x}{1+x^2}dx \quad (b)\int\frac{3x^2}{\sqrt{5x^3+9}}dx \quad (c)\int 3x^2\left(1-x^{-6}\right)dx$$

7.4. **Problem 4.** a. Find the area of the region bounded by the curves $y = x^2 - 6$ and $y = 2x + 2$.
b. A rectangular plot of land will be bounded on one side by a stream and the other three sides by a fence. If 800ft^2 of land is to be enclosed, what are the dimensions that will require the least amount of fencing? Show that your result is a minimum.

7.5. **Problem 5.** Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following:

$$(a) y = \ln\left(\cos e^x\right), \quad (b)\ y = x\tan\left(2\sqrt{x}\right) + 4 \quad (c)\ y = \int_{\sqrt{x}}^{0}\cos t^2 dt$$

7.6. **Problem 6.** a. Find the average value of the function $f(t) = 3t^2 - 2t - 4$ over the interval [1, 3].
b. Air is being pumped into a spherical balloon at a rate of 20 in^3/s. How fast is the radius increasing when the surface area is 4 in^2?

7.7. **Problem 7.** Consider the function $y = \frac{4+x^2}{x}$.
a. Find the intervals on which this function is increasing or decreasing
b. Find the intervals on which this function is concave up or concave down
c. Find all asymptotes; horizontal, vertical and/or slant
d. Determine the points (if any) at which this function has a local maximum, a local minimum or a point of inflection
e. Sketch this function making sure to label the points found in part **d**.

8. Math 111 Final Exam Solutions, Spring 2022

8.1. **Solution to Problem 1(a,b,c).**

We know that,

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1 \text{ and } 1-\cos(2x)=2\sin^2(x)$$

a.

$$\lim_{x\to 0}\frac{\sin x^2}{\cos(2x)-1}=\lim_{x\to 0}\frac{\sin x^2}{-2\sin^2(x)}=-\frac{1}{2}\lim_{x\to 0}\frac{\frac{\sin x^2}{x^2}}{\frac{\sin^2(x)}{x^2}}=-\frac{1}{2}.$$

b. Let

$$y=\left(e^{2x}+x\right)^{\frac{1}{x}},\ \text{take natural log both sides we get and taking} \lim_{x\to 0} \text{we get,}$$

$$\lim_{x\to 0}\ln(y)=\lim_{x\to 0}\frac{\ln(e^{2x}+x)}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2e^{2x}+1}{e^{2x}+1}=\frac{3}{2}.$$

Taking exponential both sides we get,

$$\lim_{x\to 0}y=e^{3/2}.$$

c.

$$\lim_{t\to -2}\frac{t^3-2t+4}{t^2-t-6}\left(\frac{0}{0}\right)=\lim_{t\to -2}\frac{3t^2-2}{2t-1}=-2.$$

8.2. **Solution to Problem 2(a,b,c).**

For problem 2(a), we need to do the implicit differentiation.

a.

$$\frac{dy}{dx}=\frac{d}{dx}(e^{x+y}-x)=e^{x+y}\left(\frac{dy}{dx}+1\right)-1$$

$$\frac{dy}{dx}=\frac{e^{x+y}-1}{1-e^{x+y}}=-1.$$

b.

$$\frac{dy}{dx}=\frac{d}{dx}x\arcsin(2x)=x\frac{d}{dx}\arcsin(2x)+\arcsin(2x)\frac{d}{dx}x=\frac{2x}{\sqrt{1-4x^2}}+\arcsin(2x)$$

c. Let

$$y=x^{1+\frac{1}{\ln(x)}},\ \text{take natural log both sides we get}$$

$$\ln(y)=\left(1+\frac{1}{\ln(x)}\right)\ln(x)=\ln(x)+1 \tag{16}$$

Differentiating equation (16) with respect to x, we get,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}\implies\frac{dy}{dx}=\frac{y}{x}=\frac{x^{1+\frac{1}{\ln(x)}}}{x}=x^{\frac{1}{\ln(x)}}. \tag{17}$$

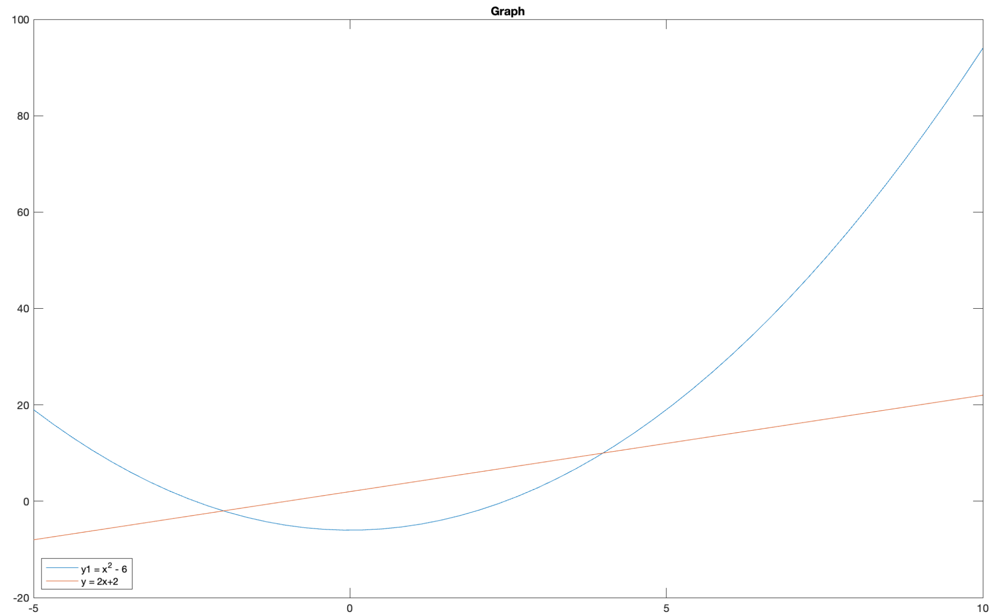

FIGURE 2.

8.3. **Solution to Problem 3(a,b,c).** Let $u = 1 + x^2 \implies du = 2xdx$

a.

$$\int_{-1}^{1} \frac{x}{1+x^2}dx = \int \frac{1}{2u}\, du = \frac{1}{2}[\ln(u)] = \frac{1}{2}[\ln\big(1+x^2\big)]_{-1}^{1} = 0.$$

b. Let $u = 5x^3 + 9 \implies du = 15x^2\, dx$

$$\int \frac{3x^2}{\sqrt{5x^3+9}}\, dx = \frac{1}{5}\frac{1}{\sqrt{u}} = \frac{2\sqrt{u}}{5} = \frac{2}{5}\sqrt{5x^3+9}.$$

c.

$$\int 3x^2(1-x^{-6})\, dx = \int 3x^2\, dx - \int \frac{3}{x^4}\, dx = x^3 + x^{-3}.$$

8.4. **Solution to Problem 4(a,b).** a. The area of the region between the curves $f(x)$and $g(x)$ in the interval $a \leq x \leq b$ is given by

$$A = \int_a^b |f(x) - g(x)|\, dx = \int_{-2}^{4} |(x^2-6) - (2x+4)|\, dx = 36.$$

8.5. **Solution to Problem 6(a,b).** a. The average rate change of the function $f(x)$ over the interval $a \leq x \leq b$ is given by

$$\frac{f(b)-f(a)}{b-a} = \frac{(3\cdot 3^2 - 2\cdot 3 - 4) - (3\cdot 1^2 - 2\cdot 1 - 4)}{3-1} = 10.$$

b. We have that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \cdot \frac{\mathrm{d}r}{\mathrm{d}t} = 20$$

which implies

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{20}{4\pi r^2}$$

We know that $A = 4\pi r^2$, and so we also have

$$4 = \frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \implies \frac{4}{8\pi r} = \frac{20}{4\pi r^2} \implies \boxed{r = 10}$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{20}{4\pi r^2} = \frac{20}{4\pi \cdot 10^2} = \frac{1}{20\pi}.$$

8.6. **Solution to 7(a, b, c, d, e).** Follow the same process mentioned in problem 7 of Exam III.

9. Common Exam I, Fall 2019

9.1. **Problem 1.** Evaluate the following limits:

$$\text{a. } \lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x+3}} \qquad \text{b. } \lim_{\theta\to 0}\frac{\tan(\pi\theta)}{\theta}.$$

9.2. **Problem 2.** Evaluate the following limit:

$$\lim_{x\to 1^-}\frac{|2-x|}{x^2-3x+2}.$$

9.3. **Problem 3.** Evaluate the following limit:

$$\lim_{t\to -3}\frac{t^2+4t+3}{t^2-9}.$$

9.4. **Problem 4.** Evaluate the following limit:

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}.$$

9.5. **Problem 5.** Evaluate the following limit:

$$\lim_{x\to +\infty}\frac{x\sqrt{x^2+4}}{2x^2+3x-6}.$$

9.6. **Problem 6.** Consider the function $y = \frac{2}{x}$.
a. Find the average rate of change of f over the interval $1 \le x \le 2$,
b. Use the definition of the derivative to find the slope of the tangent line at $x = 2$, and then find the equation of the tangent line to the curve at this point.

9.7. **Problem 7.** Find the constants a and b so that the function given below is continuous for all x:

$$f(x) = \begin{cases} x^2+3, & x < 2 \\ a, & x = 2 \\ ax+b, & x > 2. \end{cases}$$

9.8. **Problem 8.** Find an equation for all asymptotes, if they exist, for the following function, and be sure to label the type of asymptote (horizontal, vertical or oblique (slant)):
y

$$y = \frac{x^2-2x-8}{x+1}.$$

9.9. **Problem 9.** Find all points where the following function is discontinuous and identify the type of discontinuity (jump, infinite or removable). If the function has a removable discontinuity, then determine how to define f at that point in a way that extends $f(x)$ to be continuous there,

$$f = \frac{x\,|x-1|}{(x^2-x)(x-2)}.$$

9.10. **Problem 10.** Evaluate the following limit (hint: begin by multiplying numerator and denominator by the conjugate):

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sin(x)}.$$