CALCULUS II

ATUL ANURAG
EMAIL: AA2894@NJIT.EDU
WEBSITE: ATULANURAG.COM

Contents

Date: March 8, 2024.

1. Final Exam, Fall 2021

1. Evaluate the following integrals.

$$a. \int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt{x})^2} \quad b. \int \frac{4\ \mathrm{d}x}{x^2-2x}$$

2. (a) Use the Disk method to find the volume of the solid generated by revolving the region bounded between the curve $y = 2/x, y = 0, x = 1$ and $x = 4$ about the x-axis.
b. Find the length of the curve $y = (x + \frac{5}{9})^{3/2}$ from $x = 0$ to $x = 8$.
3. Evaluate the following integrals.

$$a. \int x\sin(2x)\mathrm{d}x \qquad b. \int \frac{\mathrm{d}x}{(4-x^2)^{3/2}}$$

4.(a) Find the first three terms in the Taylor series of the function $f(x) = x^3 + x$ about $a = -1$.
b. Determine if the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=0}^{\infty} \frac{2^{2n+2}}{5^n}$$

5.(a) Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \frac{n!}{(n+1)!}$$

b. Use a **comparison test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+4n^2+1}}$$

6.(a) Evaluate the following integral:

$$\int_1^{\infty} \frac{\mathrm{d}x}{x^2+3x+2}$$

b. A force of $F = \frac{x}{x^2+9}$ lbs is applied to move an object along the x-axis from $x = 0$ to $x = 4$ft. Determine the amount of work done.
7.(a) Evaluate the integral:

$$\int \sec^2(x)\ \tan(x)\ \mathrm{d}x$$

b. Determine the radius of convergence and interval of convergence for the power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{\sqrt{n+1}}$$

8.(a) Find the area of the polar region that lies inside one loop of the curve $r^2 = 4\ \sin(2\theta)$.

b. Find an equation for the line tangent to the curve $x = 4\sin(t), y = 2\cos(t)$ at the point $t = \pi/4$.

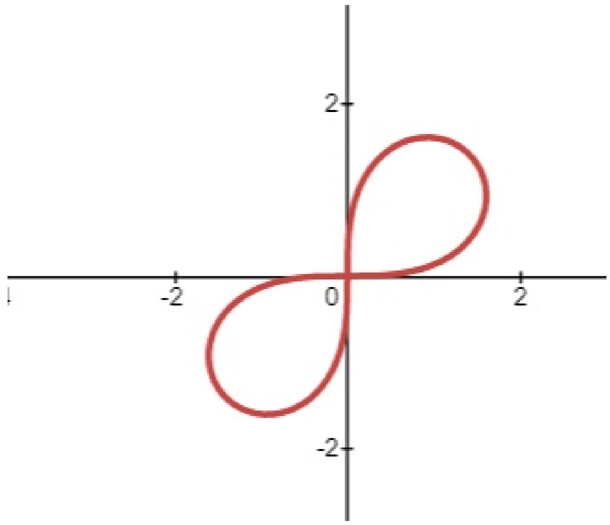

FIGURE 1. $r^2 = 4\sin(2\theta)$

2. Final Exam Solution, Fall 2021

a. Let

$$u = 1 + \sqrt{x}, \mathrm{d}u = \frac{1}{2\sqrt{x}}\mathrm{d}x$$

.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt{x})^2} &= \int \frac{2}{u^2}\mathrm{du} \\
&= -\frac{2}{u} \\
&= -\frac{2}{1+\sqrt{x}}.
\end{aligned}$$

b. Apply the partial fractions,

$$\begin{aligned}
\int \frac{4\mathrm{d}x}{x^2 - 2x} &= 4\int \frac{1}{2}\left[\frac{1}{x-2} - \frac{1}{x}\right]\mathrm{d}x \\
&= 2\ \ln\left|\frac{x-2}{x}\right| + C.
\end{aligned}$$

2a. The volume,

$$V = \int_1^4 \pi y^2 \mathrm{d}x = \int_{\substack{1\\3}}^4 \pi (2/x)^2 \mathrm{d}x = 3\pi.$$

2b. The length of the curve,

$$
\begin{aligned}
L &= \int_0^8 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x \\
&= \frac{3}{2}\int_0^8 \sqrt{1+x}\ \mathrm{d}x \\
&= 26.
\end{aligned}
$$

3a.

$$\int x\sin(2x)\,dx = \text{Let's use integration by parts with } u = x \text{ and } dv = \sin(2x)\,dx.$$

$$du = dx \quad \text{and} \quad v = -\frac{1}{2}\cos(2x).$$

Now, apply the integration by parts formula:

$$
\begin{aligned}
\int u\,dv &= uv - \int v\,du \\
&= x\left(-\frac{1}{2}\cos(2x)\right) - \int \left(-\frac{1}{2}\cos(2x)\right)\,dx \\
&= -\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx.
\end{aligned}
$$

Now, integrate $\int \cos(2x)\,dx$:

$$\int \cos(2x)\,dx = -\frac{1}{2}\sin(2x) + C,$$

where C is the constant of integration. So, the final result is:

$$
\begin{aligned}
\int x\sin(2x)\,dx &= -\frac{x}{2}\cos(2x) + \frac{1}{2}\left(-\frac{1}{2}\sin(2x)\right) + C \\
&= -\frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C.
\end{aligned}
$$

3b. Let $x = 2\ \sin(u)$,

$$\int \frac{\mathrm{d}x}{(4-x^2)^{3/2}} = \frac{x}{4}\frac{1}{\sqrt{4-x^2}} + C$$

4a. The Taylor series expansion of $f(x)$ about $a = -1$ is given by:

$$f(x) = -2 + 4(x+1) - 3(x+1)^2.$$

4b. The given series is a geometric series with $r = \frac{4}{5} < 1$. Hence, the series converges. The sum,

$$
\begin{aligned}
S = \sum_{n=0}^{\infty} \frac{2^{2n+2}}{5^n} &= 4\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n \\
&= 4\left(\frac{1}{1-\frac{4}{5}}\right) \\
&= 20.
\end{aligned}
$$

5a. Applying the ratio test,

$$
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1, \text{ the series converges.}
$$

.

5b. Applying the limit comparison test, and comparing the given series with $\sum_{n=1}^{\infty}\frac{1}{n}$. The given series diverges.

6a.

$$
\begin{aligned}
\int_1^{\infty} \frac{\mathrm{d}x}{x^2+3x+2} &= \lim_{a\to\infty}\int_1^{a}\frac{\mathrm{d}x}{x^2+3x+2} \\
&= \lim_{a\to\infty}\int_1^{a}\left[\frac{1}{x+1}-\frac{1}{x+2}\right]\mathrm{d}x \\
&= -\ln(2/3).
\end{aligned}
$$

6b. The workdone,

$$
\begin{aligned}
W = \int_0^4 F\mathrm{d}x &= \int_0^4 \frac{x}{x^2+9}\mathrm{d}x \\
&= \frac{1}{2}[\ln\left(x^2+9\right)]_0^4 \\
&= \ln(5/3).
\end{aligned}
$$

7a.

$$
\int \sec^2(x)\,\tan(x)\,\mathrm{d}x = \frac{\tan^2(x)}{2} + C
$$

7b. Using Ratio test, we get the limit,

$$
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1|.
$$

The series converges whenever

$$
|x-1| < 1.
$$

Therefore, **The radius of converges**, R=1.

Interval of convergence is where $|x-1| < R \Rightarrow |x-1| < 1 \Rightarrow 0 < x < 2$. Now we are also required to check the endpoints $x = 0, x = 2$.

Case 1: When $x = 0$, the above series is

$$
\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}
$$

Using p-series test, the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$$

diverges.

Case 2: When $x = 2$, the above series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

Using Alternating series test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

converges.

Therefore, **Interval of convergence** is $0 < x \leq 2$.

8a. The area of the polar region,

$$A = \frac{1}{2}\int_0^{\pi/2} 4\ \sin(2\theta)\mathrm{d}\theta = [-\cos(2\theta)]_0^{\pi/2} = 2.$$

8b. Equation for the tangent line:

$$(y - y_0) = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} (x - x_0),$$

where:

$$x_0 = 4\sin(\pi/4) = \frac{4}{\sqrt{2}} = 2\sqrt{2},$$
$$y_0 = 2\cos(\pi/4) = \sqrt{2},$$
$$\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = \left(\frac{-2\sin(t)}{4\cos(t)}\right)_{t=\pi/4} = -\frac{1}{2}.$$

Therefore, the equation for the tangent line is:

$$\boxed{(y - \sqrt{2}) = -\frac{1}{2}(x - 2\sqrt{2})}.$$

3. Common Exam III Solutions, Spring 2022

1. Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \frac{2+n^2}{1+2n^2}$$

Sol. Here

$$a_n = \frac{2+n^2}{1+2n^2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n^2}{1+2n^2} = \lim_{n\to\infty} \frac{(\frac{2}{n^2}+1)}{(\frac{1}{n^2}+2)}$$
$$= \frac{1}{2} \neq 0$$

So, by **n-th term test** the series

$$\sum_{n=1}^{\infty} \frac{2+n^2}{1+2n^2}$$

diverges.

b. Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}}$$

Sol. Here

$$a_n = \frac{1}{n^2+\sqrt{n}}.$$

Let $b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
Find $\lim_{n\to\infty} \frac{a_n}{b_n}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Since,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges },$$

$$\textbf{By comparison test } \sum_{n=1}^{\infty} a_n \text{ converges }.$$

2. Find the sum of the following series:

$$(a) \sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \qquad (b) \sum_{n=0}^{\infty} \frac{2^{2n+1}-6^n}{8^n}$$

Sol.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= \lim_{N\to\infty} \left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)\right]$$
$$= \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right)$$
$$= 1.$$

Sol.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2^{2n+1}-6^n}{8^n}=\sum_{n=0}^{\infty}\frac{2\times 4^n-6^n}{8^n}&=2\sum_{n=0}^{\infty}\frac{4^n}{8^n}-\sum_{n=0}^{\infty}\frac{6^n}{8^n}\\&=2\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\\&=4-4\\&=0.\end{aligned}$$

Note: for 2(b) we are using **geometric series** sum:

$$\sum_{n=1}^{\infty}r^n=\frac{1}{(1-r)},\,|r|<1.$$

Question 3(a): Use the **integral test** to determine whether the series converges or diverges:

$$\sum_{n=3}^{\infty}\frac{1}{n(\ln(n)+1)^2}$$

Solution 3(a): Let

$$f(x)=\frac{1}{x(\ln(x)+1)^2}$$

(1) $f(x)$ is a continuous and non-negative function on $(3,\infty)$,
(2) $f(x)$ is a decreasing function on $(3,\infty)$, and

$$\begin{aligned}\int_3^{\infty}\frac{1}{x(\ln(x)+1)^2}\,dx \xrightarrow[du=\frac{1}{x}dx]{u=\ln(x)+1}\int\frac{1}{u^2}\,du&=-\frac{1}{u}\\&=-\left[\frac{1}{\ln(x)+1}\right]_3^{\infty}\\&=\lim_{b\to\infty}-\left[\frac{1}{\ln(x)+1}\right]_3^{b}\\&=\lim_{b\to\infty}\left[-\frac{1}{\ln(b)+1}+\frac{1}{\ln(3)+1}\right]\\&=\frac{1}{\ln(3)+1}.\end{aligned}$$

Since the limit of the integral is finite. By **integral test**, the series

$$\sum_{n=3}^{\infty}\frac{1}{n(\ln(n)+1)^2}$$

converges.

Question 3(b): Use a **comparison test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}+\left(\frac{1}{2}\right)^n\right)$$

Solution 3(b): Let $b_n = \sum_{n=1}^{\infty} \frac{1}{n}$. We have,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1 + \frac{n}{2^n}\right) = 1, \text{ and } \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

$$\textbf{By comparison test } \sum_{n=1}^{\infty} a_n \text{ diverges .}$$

Question 4(a): Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$$

Solution 4(a): Let's apply the ratio test to the given series:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{e^{(n+1)^2}}{(n+1)!}}{\frac{e^{n^2}}{n!}} \right| = \lim_{n\to\infty} \left| \frac{e^{(n+1)^2} \cdot n!}{e^{n^2} \cdot (n+1)!} \right|$$

Now, simplify the expression inside the absolute value:

$$L = \lim_{n\to\infty} \left| \frac{e^{(n+1)^2} \cdot n!}{e^{n^2} \cdot (n+1)!} \right| = \lim_{n\to\infty} \left| \frac{e^{(n+1)^2}}{e^{n^2}} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| e^{(n+1)^2 - n^2} \cdot \frac{1}{n+1} \right|$$

Simplify further:

$$L = \lim_{n\to\infty} \left| e^{(n+1)^2 - n^2} \cdot \frac{1}{n+1} \right| = \lim_{n\to\infty} \left| e^{2n+1} \cdot \frac{1}{n+1} \right| = \lim_{n\to\infty} \left| \frac{e^{2n+1}}{n+1} \right| = \infty$$

Since $\lim_{n\to\infty} L = \infty$, by the ratio test, the series $\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$ diverges.

Question 4(b): Use the **root test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \left(\frac{1}{3} + \frac{1}{n}\right)^{2n}$$

Solution 4(b): Here, $a_n = \left(\frac{1}{3} + \frac{1}{n}\right)^{2n}$, and

$$\begin{aligned} \lim_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\left(\frac{1}{3} + \frac{1}{n}\right)^{2n}\right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{n}\right)^{2} \\ &= \frac{1}{9} < 1 \end{aligned}$$

By **ratio test** the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3} + \frac{1}{n}\right)^{2n}$$

converges.

Question 5(a): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{\sqrt{n^6+n}}$$

Solution 5(a): Here $a_n = (-1)^n \frac{n^2}{\sqrt{n^6+n}}$, $|a_n| = \frac{n^2}{\sqrt{n^6+n}}$, let $b_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^5}}} = 1$$

Since the series, $\sum_{n=1}^{\infty} b_n$ diverges(by p-test). By **comparison test**, the series $\sum_{n=1}^{\infty}|a_n|$ diverges. Therefore, the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{\sqrt{n^6+n}}$$

is not absolutely convergent.

Using the alternating series test on the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{\sqrt{n^6+n}} = \sum_{n=1}^{\infty}(-1)^n a_n,$$

we have $a_n = \frac{n^2}{\sqrt{n^6+n}}$.

(1) $a_{n+1} < a_n$, for all $n \geq 1$,
(2) $a_n > 0$, for all $n \geq 1$,
(3) $\lim_{n\to\infty} a_n = 0$.

Therefore, by **alternating series test** the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{\sqrt{n^6+n}}$$

converges **conditionally.**

Question 5(b): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{4^n+2}$$

Solution 5(b): Here, we have the series $a_n = \sum_{n=1}^{\infty}(-1)^n \frac{2^n}{4^n+2}$ and $|a_n| = \sum_{n=1}^{\infty} \frac{2^n}{4^n+2}$.
Let's define another series

$$b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

for comparison.
Since, $|a_n| \leq b_n$, and the series, $\sum_{n=1}^{\infty} b_n$ converges(geometric series) .
By **Direct comparison test**, the series $\sum_{n=1}^{\infty}|a_n|$ converges. Therefore, the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{4^n+2}$$

is absolutely convergent.
Question 6: Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2 \ 2^n}$$

Solution 6: Using Ratio test, we get the limit,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x+3|}{2}.$$

The series converges whenever

$$\frac{|x+3|}{2} < 1.$$

Therefore, **The radius of converges**, R=2.
Interval of convergence is where

$$|x+3| < R \Rightarrow |x+3| < 2 \Rightarrow -5 < x < -1.$$

Now we also need to check the endpoints $x = -5, x = -1$
Case 1: When $x = -5$, the above series is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2 \ 2^n} &= \sum_{n=1}^{\infty} \frac{(-5+3)^n}{n^2 \ 2^n} \\ &= \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 \ 2^n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}. \end{aligned}$$

Using alternating series test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges.
Case 2: When $x = -1$, the above series is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2 \ 2^n} &= \sum_{n=1}^{\infty} \frac{(-1+3)^n}{n^2 \ 2^n} \\ &= \sum_{n=1}^{\infty} \frac{(2)^n}{n^2 \ 2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Using **p-test**, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Therefore, **Interval of convergence** is

$$-5 \le x \le -1.$$

Question 7: Find the first 3 non-zero terms in the Taylor series about $a = 1$ for the function $f(x) = 3 - x + 2x^2$.
Solution 7: The Taylor series expansion about $a = 1$ given by,

$$f(x) = f(1) + (x-1)\ f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots,$$

where

$$f(1) = 4, f'(1) = 3, f''(1) = 4, f^{(n)} = 0(\text{ for all } \ n \ge 3).$$

Therefore,

$$f(x) = 4 + 3(x-1) + \frac{4\ (x-1)^2}{2!} = 4 + 3(x-1) + 2(x-1)^2.$$

Question 8: Write down the first 3 non-zero terms in the Maclaurin series for the function $f(x) = e^{-x}\sin(2x)$.
Solution 8: The Maclaurin series expansion is given by,

$$f(x) = e^{-x}\sin(2x) = [1 - x + \frac{x^2}{2} + \cdots][2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots].$$

Comparing the cofficients of $x^0, x^1, x^2, \cdots$, we have,

$$f(x) = 2x - 2x^2 - \frac{1}{3}x^3 + \cdots.$$

Question 9: Solve for x

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots = 2$$

Solution 9: Observe,

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots = e^{-x}.$$

Therefore, we have,

$$e^{-x} = 2.$$

Solving for x, we get,

$$x = -\ln(2).$$

4. Problem Set I

1. Determine the value of

$$\int_0^1 x^2 \cos(x) \mathrm{d}x.$$

2. Find the area of the surface obtained by rotating the curve

$$y = x^3, \ 0 \leq x \leq 2,$$

about the x-axis.

3. Determine whether the series

$$2 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{4}} + \cdots$$

converges absolutely, converges conditionally, or diverges.

4. Evaluate the integral

$$\int_0^1 \frac{x^2}{(x^3+1)^2} \ \mathrm{d}x$$

5. Let $F(x) = \int_1^x \ln(t) \ \mathrm{d}t$. Find the value value $F''(2)$.

6. Consider the region bounded by the graphs of $f(x) = x^2 + 1$ and $g(x) = 3 - x^2$. Write the integral for the volume of the solid of revolution obtained by rotating this region about the x-axis. Do not evaluate the integral.

7. Evaluate the integral

$$\int \frac{x+1}{x^2(x-1)} \ \mathrm{d}x$$

8. Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7 + n^2}}$$

9. Use Simpson's rule with n = 6 to estimate

$$\ln(3) = \int_1^3 \frac{1}{x} \ \mathrm{d}x$$

10. Determine whether the following integral converges, and if so, evaluate it.

$$\int_0^{\infty} \cos(x) \ \mathrm{d}x$$

11. Find the length of the curve $x = \frac{1}{3}(y^2+2)^{3/2}$, from $y = 0$ to $y = 3$.

12. Determine whether the series converges or diverges. If it converges then find its sum.

$$\sum_{n=0}^{\infty} \left(\frac{5}{4^n} + \frac{(-1)^{n+1}}{3^n} \right)$$

13. Find the volume of the solid of revolution formed by revolving the y-axis, the region enclosed by

$$y = \cos(x^2),$$

and the $x-$axis.

14. Find the following approximations to

$$\int_0^{\pi/2} \cos(x)\ \mathrm{d}x$$

a. Using the **trapezoidal rule** with two intervals.
b. Using **Simpson's rule** with two intervals.

5. Solutions to Problem Set I

1.

$$\begin{aligned}
\int_0^1 x^2\cos(x)\mathrm{d}x &= x^2\int_0^1 \cos(x)\mathrm{d}x - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}x^2 \int \cos(x)\mathrm{d}x \\
&= x^2\sin(x)\Big|_0^1 - \int_0^1 2x\sin(x)\mathrm{d}x \\
&= \sin(1) - 2\Bigg[\int_0^1 x\sin(x)\mathrm{d}x\Bigg] \\
&= \sin(1) - 2\Bigg[x\int_0^1 \sin(x) - \int \frac{\mathrm{d}}{\mathrm{d}x}x\int \sin(x)\mathrm{d}x\Bigg] \\
&= \sin(1) - 2\Bigg[-x\cos(x)\Big|_0^1 + \int_0^1 \cos(x)\mathrm{d}x\Bigg] \\
&= \sin(1) - 2\Bigg[-\cos(1) + \sin(x)\Big|_0^1\Bigg] \\
&= \sin(1) - 2\Bigg[-\cos(1) + \sin(1)\Bigg] \\
&= 2\cos(1) - \sin(1).
\end{aligned}$$

2. The area of the surface,

$$\begin{aligned}
S &= 2\pi\int_0^2 y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ \mathrm{d}x \\
&= 2\pi\int_0^2 x^3\sqrt{1+9x^4}\ \mathrm{d}x \\
&= \frac{\pi}{27}\Big[(1+9x^4)^{\frac{3}{2}}\Big]_0^2
\end{aligned}$$

We integrate the above integral by substituting

$$u = 1 + 9x^4.$$

3. To determine whether the series

$$2 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{4}} + \cdots$$

converges absolutely, converges conditionally, or diverges, we need to examine the convergence of both the original series and the absolute value of the series.

The Alternating Series Test states that if a series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ satisfies two conditions:

(1) a_n is positive (for all n).

(2) $a_{n+1} \leq a_n$ (i.e., the terms are decreasing in absolute value).

Then the series converges.

In this case, the terms are $a_n = \frac{2}{\sqrt{n}}$, and a_n is positive and decreasing for all n. So, the original series converges by the Alternating Series Test.

2. Absolute Value of the Series: Now, let's consider the absolute value of the series:

$$\sum_{n=1}^{\infty}\left|2 - \frac{2}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\left|2\left(1 - \frac{1}{\sqrt{n}}\right)\right| = 2\sum_{n=1}^{\infty}\left(1 - \frac{1}{\sqrt{n}}\right)$$

The series $\sum_{n=1}^{\infty} 1$ is a simple harmonic series, which is known to diverge.

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is also a divergent p-series with $p = \frac{1}{2}$, and p is less than 1. Therefore, it also diverges.

Since the absolute value of the series diverges, we can conclude that the original series also diverges.

- The original series converges (by the Alternating Series Test).
- The absolute value of the series diverges.

Therefore, the original series converges conditionally.

4. To evaluate the integral

$$\int_0^1 \frac{x^2}{(x^3+1)^2}\,\mathrm{d}x$$

Let $u = x^3 + 1$, which implies $\mathrm{d}u = 3x^2\,\mathrm{d}x$.

Rearrange this equation to solve for $\mathrm{d}x$:

$$\mathrm{d}x = \frac{1}{3x^2}\,\mathrm{d}u$$

Now, substitute u and $\mathrm{d}u$ back into the integral:

$$\int_0^1 \frac{x^2}{(x^3+1)^2}\,\mathrm{d}x = \int_1^2 \frac{1}{3u^2}\,\mathrm{d}u$$

Next, we can pull the constant $\frac{1}{3}$ out of the integral:

$$\frac{1}{3}\int_1^2 \frac{1}{u^2}\,\mathrm{d}u = \frac{1}{3}\left[-\frac{1}{u}\right]_1^2 = \frac{1}{3}\left(-\frac{1}{2} + \frac{1}{1}\right) = \frac{1}{6}.$$

So, the value of the integral

$$\int_0^1 \frac{x^2}{(x^3+1)^2}\,dx = \frac{1}{6}.$$

5. Given, $F(x) = \int_1^x \ln(t)\ \mathrm{d}t \implies F'(x) = \ln(x) \implies F''(x) = -\frac{1}{x^2}$.

Therefore,

$$F''(2) = -\frac{1}{4}.$$

6. To find the volume of the solid of revolution obtained by rotating the region bounded by the graphs of $f(x) = x^2 + 1$ and $g(x) = 3 - x^2$ about the x-axis, you can use the method of cylindrical shells. The formula for the volume of a solid of revolution using cylindrical shells is:

$$V = 2\pi \int_a^b x \cdot |(f(x) - g(x))| \, dx$$

In this case, a and b are the points of intersection between the two curves $f(x)$ and $g(x)$,

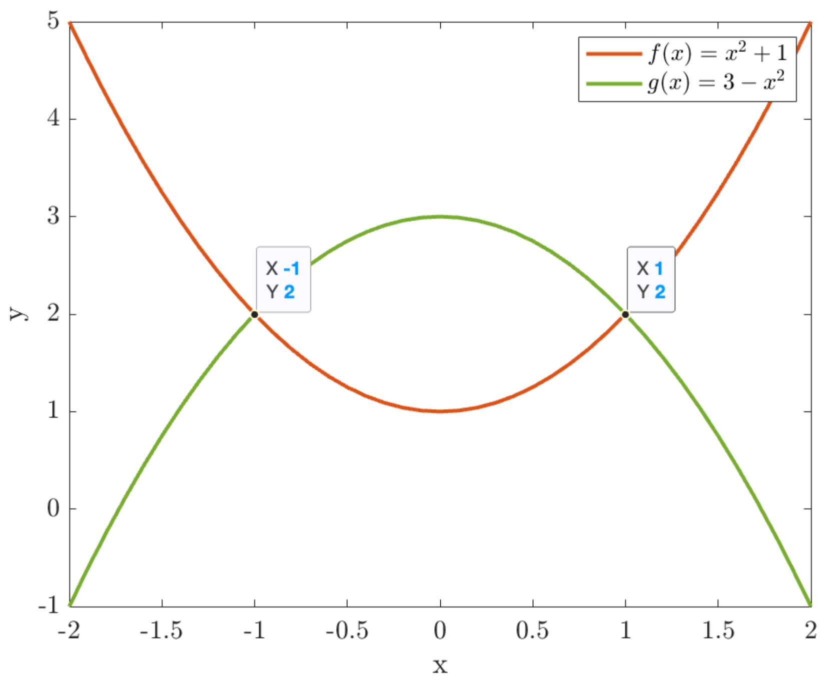

FIGURE 2. Plot depicting the intersection of $f(x)$ and $g(x)$

which you can find by setting them equal to each other:

$$x^2 + 1 = 3 - x^2$$

Solving for x:

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = \pm 1$$

So, $a = -1$ and $b = 1$.

Now, you can set up the integral for the volume:

$$\begin{aligned} V &= 2\pi \int_{-1}^{1} x \cdot |(x^2+1) - (3-x^2)|\, dx \\ &= 4\pi \int_{-1}^{1} x \cdot |2x^2 - 2|\, dx. \end{aligned}$$

11. Given, $x = \frac{1}{3}(y^2+2)^{3/2} \implies \frac{\mathrm{d}x}{\mathrm{d}y} = y\sqrt{y^2+2}$.

The length of the curve,

$$\begin{aligned} L &= \int_0^3 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y \\ &= \int_0^3 \sqrt{1 + (y\sqrt{y^2+2})^2}\ \mathrm{d}y \\ &= \int_0^3 \sqrt{y^4 + 2y^2 + 1}\ \mathrm{d}y \\ &= \int_0^3 y^2 + 1\ \mathrm{d}y \\ &= 12. \end{aligned}$$

13. The volume of a solid of revolution using cylindrical shells is:

$$V = 2\pi \int_a^b x f(x)\, \mathrm{d}x$$

In this case, the limits of integration a and b correspond to the x-values where the curve $y = \cos(x^2)$ intersects the x-axis. To find these points, set y equal to zero and solve for x:

$$\cos\left(x^2\right) = 0$$

This occurs when $x^2 = \frac{\pi}{2} + n\pi$ for n being an integer. Therefore, the points of intersection are given by:

$$x = \pm\sqrt{\frac{\pi}{2} + n\pi}$$

Now, we need to determine the appropriate range of integration. Since $\cos(x^2)$ is an even function, it's sufficient to find the volume for x in the positive range (i.e., from 0 to the first positive intersection point), and then multiply it by 2 to account for the negative side. So, we will integrate from $x = 0$ to $x = \sqrt{\frac{\pi}{2}}$.

Now, we can set up the integral:

$$
\begin{aligned}
V &= 2\pi \int_0^{\sqrt{\frac{\pi}{2}}} x\cos\bigl(x^2\bigr)\,\mathrm{d}x \\
&= 2\pi \int \frac{\cos(u)}{2}\,\mathrm{d}u \\
&= \pi\sin(u) \\
&= \pi\sin\bigl(x^2\bigr)\Big|_0^{\sqrt{\frac{\pi}{2}}} \\
&= \pi.
\end{aligned}
$$

6. Problem Set II

Problem 1: Find the Taylor series for e^{-x^2} centered at 0. What is the interval of convergence for this series?

Problem 2: Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{e^{-n}}{n^2 + 2n}$$

Problem 3: Determine whether the following series converge or diverge

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Problem 4: Determine whether the following series converge or diverge

$$\sum_{n=1}^{\infty} \frac{n + 4^n}{n + 6^n}$$

Problem 5: For each of the following power series, find the interval of convergence and the radius of convergence:

$a.$ $\sum_{n=1}^{\infty} (-1)^n n^2 x^n$

$b.$ $\sum_{n=1}^{\infty} \frac{2^n}{n^2} (x-3)^n$

$c.$ $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n!} (x-10)^n$

Problem 6: Consider the function $g(x)$ defined by the power series:

$$g(x) = \sum_{n=0}^{\infty} \frac{2^n (n!)^2 x^n}{(2n)!}$$

$a.$ Find the radius of convergence of the power series.

$b.$ Use the first 3 non-zero terms of the power series to estimate

$$\int_0^1 \frac{g(x) - 1}{x} \, \mathrm{d}x$$

Problem 7: determine whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

Problem 8: Find the Maclaurin series for $f(x) = \frac{1}{1+2x^2}$. What is the interval of convergence for this series?

7. Problem Set III

Problem 1: Find the first three terms in the Taylor Series of the function $f(x) = x^3 + x,$ at $a = -1$.

Problem 2: Determine if the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=0}^{\infty} \frac{2^{2n+2}}{5^n}$$

Problem 3: Use any test to check the convergence or divergence of the given series.

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \frac{n!}{(n+1)!}$$

Problem 4: Use any test to check the convergence or divergence of the given series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ (x-1)^n}{\sqrt{n+1}}$$

Problem 5: Find the power series representation for $\ln(1-x)$ and its radius of convergence.

Problem 6: Find the first three nonzero terms in the Maclaurin series for $f(x) = e^x \ \sin(x)$.

Problem 7: Use any test to check the convergence or divergence of the given series.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

8. Problem Set IV

Problem 1: Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$(a)\ \sum_{n=2}^{\infty} \frac{(-1)^n}{\text{ln n}}$

$(b)\ \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

$(c)\ \sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

Problem 2: Find the **radius of convergence and interval of convergence** of the series.

$(a)\ \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

$(b)\ \sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

$(c)\ \sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n,\ b > 0$

3(a) Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\ 9^n}$$

3(b) Use the **root test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{3n}\right)^n$$

4(a): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{4n^3 + n}}$$

4(b) Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{e^{2n} + 1}$$

5(a) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \left(\frac{2 + n^2}{1 + 2n^2}\right)^n$$

5(b) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{1/n}$$

6(a) Write down the first 3 non-zero terms in the Maclaurin series for the function $f(x) = x + \cos(2x)$.

6(b) Find the first 3 non-zero terms in the Taylor series about $a = 1$ for the function $f(x) = 2 - x^2$.

7. Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\ 3^n}$$

8(a): Solve for x

$$1 + x + x^2 + x^3 + \cdots = 2$$

8(b): Find the Taylor polynomial or order 2 generated by $f(x) = \ln(x)$ about $a = 1$.

9. Problem Set V

Problem 1: Find the value of p for which the series is convergent.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln\ \mathrm{n})^{\mathrm{p}}}$$

Problem 2: Find the value of p for which the series is convergent.

$$\sum_{n=3}^{\infty} \frac{1}{n \ln(\ln(\ln\ n))^{p}}$$

Problem 3: Find the value of p for which the series is convergent.

$$\sum_{n=1}^{\infty} \frac{\ln\ n}{n^{p}}$$

Problem 4: Find the interval of convergence for $f, f',$ and f''.

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Problem 5: Find the value of p for which the series is convergent.

$$\sum_{n=1}^{\infty} n(1+n^2)^{p}$$

Problem 6: Find the value of the integral:

$$\int_0^{1/2} \frac{\mathrm{d}x}{x^2 - x + 1}$$

Problem 7: Find the sum of the following series.

$(i) \sum_{n=2}^{\infty} n(n-1)x^n, |x|< 1,$ $\quad (ii) \sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$ $\quad (iii) \sum_{n=1}^{\infty} \frac{n^2}{2^n}$

10. Problem Set VI

Problem 1: Apply the root test to the following series.

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+3} \right)^n$$

Problem 2: Apply the root test to the following series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{2n}}$$

Problem 3: Apply the ratio test to the following alternating series.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{1000^n}$$

Problem 4: Evaluate the following series using any test.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

Problem 5: Evaluate the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$$

Problem 6: Evaluate the following series converges or diverges.

$$\sum_{n=1}^{\infty} (0.8)^{-n} \ n^{-0.8}$$

Problem 7: Evaluate the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

Problem 8: State whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

11. Problem Set VII

1. Integrate

$$\int_0^\infty \frac{1}{1+x^2}\ \mathrm{d}x$$

2. Integrate

$$\int \frac{x^3}{\sqrt{1-x^2}}\ \mathrm{d}x$$

3. Integrate

$$\int \frac{1}{(4x^2+9)^2}\ \mathrm{d}x$$

4. Integrate

$$\int x\ \tan^{-1}(x)\ \mathrm{d}x$$

12. Problem Set VIII

Q1 Find the following integrals:

(a) $\displaystyle\int \frac{3x^2-6}{(x-1)^3}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{x^4-2x^3+4x^2+4}{x^3+x}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{x^3}{(4x^2+1)^{\frac{3}{2}}}\,\mathrm{d}x$

(e) $\displaystyle\int \frac{2\arcsin(x)}{x^3}\,\mathrm{d}x$

13. Solutions to Problem Set VIII

1a.

$$\begin{aligned}\int \frac{3x^2-6}{(x-1)^3}\,\mathrm{d}x &= 3\int \frac{x^2-2}{(x-1)^3}\,\mathrm{d}x\\ &= 3\int \left(\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right)\mathrm{d}x\\ &= 3\left(-\frac{1}{x-1}+\frac{1}{(x-1)^2}\right)+C\end{aligned}$$

1b.

$$\begin{aligned}\int \frac{\sqrt{x^2-9}}{x}\,\mathrm{d}x &= \int \frac{\sqrt{x^2-9}}{x}\left(\frac{x}{\sqrt{x^2-9}}\right)\mathrm{d}x\\ &= \int \frac{x^2-9}{x^2}\,\mathrm{d}x\\ &= \int\left(1-\frac{9}{x^2}\right)\mathrm{d}x\\ &= x-9\ln|x|+C\end{aligned}$$

1c.

$$\begin{aligned}\int \frac{x^4-2x^3+4x^2+4}{x^3+x}\,\mathrm{d}x &= \int\left(\frac{x(x^3-2x^2)}{x(x^2+1)}+\frac{4}{x(x^2+1)}\right)\mathrm{d}x\\ &= \int\left(\frac{x^2(x-2)}{x^2+1}+\frac{4}{x(x^2+1)}\right)\mathrm{d}x\\ &= \int\left(x-2-\frac{2}{x^2+1}\right)\mathrm{d}x\\ &= \frac{x^2}{2}-2x-2\arctan(x)+C\end{aligned}$$

1d.

$$\begin{aligned}\int \frac{x^3}{(4x^2+1)^{\frac{3}{2}}}\,\mathrm{d}x &= \frac{1}{8}\int \frac{8x^3}{(4x^2+1)^{\frac{3}{2}}}\,\mathrm{d}x \\ &= \frac{1}{8}\int \frac{d}{dx}\left(\frac{1}{\sqrt{4x^2+1}}\right)\,\mathrm{d}x \\ &= \frac{1}{8}\left(\frac{1}{\sqrt{4x^2+1}}\right)+C\end{aligned}$$

1e.

$$\begin{aligned}\int \frac{2\arcsin(x)}{x^3}\,\mathrm{d}x &= 2\int \frac{\arcsin(x)}{x^3}\,\mathrm{d}x \\ &= -\frac{2\arcsin(x)}{2x^2} - 2\int \left(\frac{d}{dx}\frac{\arcsin(x)}{x^2}\right)\,\mathrm{d}x \\ &= -\frac{\arcsin(x)}{x^2} + 2\int \frac{1}{x^2\sqrt{1-x^2}}\,\mathrm{d}x \\ &= -\frac{\arcsin(x)}{x^2} + 2\int \frac{d}{dx}\left(-\frac{1}{\sqrt{1-x^2}}\right)\,\mathrm{d}x \\ &= -\frac{\arcsin(x)}{x^2} - 2\left(-\frac{1}{\sqrt{1-x^2}}\right)+C \\ &= \frac{\arcsin(x)}{x^2} + \frac{2}{\sqrt{1-x^2}}+C\end{aligned}$$

14. Problem Set IX

1. Determine whether the following sequences $\{a_n\}$ are convergent or divergent. Find the limit of any convergent sequences.

$$\textbf{a. } a_n = \left(\frac{n^3 + 5n^4}{2n^4 + 2n - 1}\right)^{\frac{1}{3}} \qquad \textbf{b. } a_n = n \ \sin\left(\frac{1}{n}\right)$$

2. Evaluate

$$\int \frac{1}{(1+x^2)^{\frac{5}{2}}}\ \mathrm{d}x$$

3. Check the convergence or divergence of the following sequences:

$$\textbf{a. } a_n = \log(2n^2+1) - 2\ \log(n) \qquad \textbf{b. } a_n = ne^{-n}$$

4. Check the convergence or divergence of the following integral:

$$\int_1^\infty \frac{1}{(x^2+3x+2)}\ \mathrm{d}x$$

5. Evaluate

$$\int \ln\big(x^2+1\big)\ \mathrm{d}x$$

6. Check the convergence or divergence of the following sequences:

$$\textbf{a. } a_n = \log(n^2+1) - 3\ \log(n) \qquad \textbf{b. } a_n = \sqrt{n}\left(1 - \cos\left(\frac{1}{n}\right)\right)$$

7. Consider the integral

$$\int_2^4 \frac{1}{2x-3}\ \mathrm{d}x.$$

Estimate the integral using the **trapezoidal** rule with $n = 4$ steps.

8. Evaluate the following integrals if they are convergent or show they are divergent:

$$\textbf{a. } \int_0^\pi \tan^2(x)\ \sec^2(x)\ \mathrm{d}x \qquad \textbf{b. } \int_1^\infty \frac{x}{x^2+1}\ \mathrm{d}x$$

15. Common Exam II Solutions, Spring 2022

Solution 1. Determine if the following sequences converge or diverge. If they converge, find the limit.

a.

$$
\begin{aligned}
a_n = \frac{\sqrt{n+4n^2}}{2+n} &= \left[\frac{\sqrt{\frac{1}{n}+4}}{\frac{2}{n}+1}\right]_{n\to\infty} \\
&= \sqrt{4} \\
&= 2
\end{aligned}
$$

b.

$$
\begin{aligned}
a_n &= 2^{\frac{1}{n}} \\
\ln(a_n) &= \ln(2)^{\frac{1}{n}} \\
\ln(a_n) &= \frac{1}{n}\ln(2)
\end{aligned}
$$

Applying the limit to both sides and taking the exponential, we get:

$$\lim_{n\to\infty} a_n = 1.$$

Solution 2. Evaluate the integrals:

a.

$$
\begin{aligned}
\int \cos^3(\theta)\mathrm{d}\theta &= \int \cos(\theta)\cos^2(\theta)\mathrm{d}\theta \\
&= \int \cos(\theta)(1-\sin^2(\theta))\mathrm{d}\theta \\
&= \int \cos(\theta)\mathrm{d}\theta - \int \cos(\theta)\sin^2(\theta)\mathrm{d}\theta \\
&= \sin(\theta) - \frac{\sin^3(\theta)}{3} + C
\end{aligned}
$$

b. Integrate by parts:

$$
\begin{aligned}
\int \frac{\ln(x)}{x^2}\mathrm{d}x &= \ln(x)\int \frac{1}{x^2}\mathrm{d}x - \int \frac{d}{dx}(\ln(x))\int \frac{1}{x^2}\mathrm{d}x \\
&= -\frac{\ln(x)}{x} + \int \frac{1}{x^2}\mathrm{d}x \\
&= -\left[\frac{\ln(x)}{x} + \frac{1}{x}\right] + C
\end{aligned}
$$

Solution 3: Evaluate the integrals:

a.

$$\int \sqrt{4-x^2}\,\mathrm{d}x \Rightarrow \mathrm{d}x = 2\cos(u)\,\mathrm{d}u, \quad x = 2\sin(u)$$

$$4\int \cos^2(u)\,\mathrm{d}u = 2\int (1+\cos(2u))\,\mathrm{d}u = 2\left[u + \frac{\sin(2u)}{2}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\frac{x}{2}\right)\right] + C$$

b.

$$\int (1+\tan(x))\cos(x)\,\mathrm{d}x = \int (\cos(x)+\sin(x))\,\mathrm{d}x = -\sin(x)+\cos(x)+C$$

4a.

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-16}} \Rightarrow \mathrm{d}x = 4\sec(u)\tan(u)\mathrm{d}u, \quad x = 4\sec(u)$$

$$\int \frac{\sec(u)\tan(u)\mathrm{d}u}{\sqrt{\sec^2(u)-1}} = \int \sec(u)\,\mathrm{d}u = \ln|\tan(u)+\sec(u)|+C$$

4b.

$$\int \frac{3x-5}{x^2-3x+2}\,\mathrm{d}x = \int \left(\frac{1}{x-2} + \frac{2}{x-1}\right)\mathrm{d}x = \ln|x-2|+2\ln|x-1|+C$$

5(a)

$$\int \sec^3(\theta)\ \tan(\theta)\ \mathrm{d}\theta \Rightarrow_{\substack{u=\sec(\theta)\\ \mathrm{d}u=\sec(\theta)\tan(\theta)\ \mathrm{d}\theta}} \int u^2\ \mathrm{d}u = \frac{u^3}{3} = \frac{\sec^3(\theta)}{3} + C$$

5(b)

$$\int \frac{x}{x^4+1}\,\mathrm{d}x = \int \frac{x}{(x^2)^2+1}\,\mathrm{d}x \Rightarrow_{\substack{u=x^2\\ \mathrm{d}u2x\mathrm{d}x}}$$

$$= \frac{1}{2}\int \frac{1}{u^2+1}\,\mathrm{d}u$$

$$= \frac{1}{2}\tan^{-1}(u)$$

$$= \frac{1}{2}\tan^{-1}(x^2) + C$$

6.(a) Apply the Direct comparison test to determine if the following integral converges.

$$\int_0^1 \frac{1}{x^2+\sqrt{x}}\,\mathrm{d}x \le \int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to 0} 2\left[\sqrt{x}\right]_a^1 = 2,\ \text{convergent}$$

6(b). Apply the Limit Comparison test to determine if the following integral converges.

$$\int_1^\infty \frac{e^x}{x\sqrt{e^{2x}+4}}$$

$$\lim_{x\to\infty} \frac{\frac{e^x}{x\sqrt{e^{2x}+4}}}{1/x} = 1,\ \text{and}\ \int_1^\infty \frac{1}{x}\mathrm{d}x = \textbf{Diverges},\ \textbf{therefore above integral diverges.}$$

7. Find the step size,

$$\boxed{h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}}$$

$$\boxed{x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2}$$

$$\boxed{f(0) = -1,\ f(1/2) = -7/8,\ f(1) = 0,\ f(3/2) = 19/8,\ f(2) = 7}$$

Using Trapezoidal rule:

$$\int_0^2 (x^3 - 1)\ \mathrm{d}x = \frac{h}{2}\left[f(0) + 2\Big(f(1/2) + f(1) + f(3/2)\Big) + f(2)\right] = \frac{9}{4}.$$

Error bound,

$$E = \frac{\max|f'''(x)|(b-a)^3}{12n^2} = \frac{6.(2-0)^3}{12.4^2} = \frac{1}{4}.$$

8. Evaluate the improper integrals:

a.

$$\int_0^\infty \frac{e^{-x}}{1+e^{-x}}\ \mathrm{d}x = \lim_{a\to\infty}\int_0^a \frac{e^{-x}}{1+e^{-x}}\ \mathrm{d}x \Rightarrow_{\substack{u=1+e^{-x}\\ \mathrm{d}u=-e^{-x}\mathrm{d}x}} = \lim_{a\to\infty}\Big[-\ln|1+e^{-x}|\Big]_0^a = \ln(2).$$

b.

$$\int_1^2 \frac{1}{\sqrt{x-1}}\ \mathrm{d}x = \lim_{a\to 1}\int_a^2 \frac{1}{\sqrt{x-1}}\ \mathrm{d}x = \lim_{a\to 1}\Big[2\sqrt{x-1}\Big]_a^2 = 2.$$

9.

$$\int_0^{\pi/4} \frac{\sec^4(x)}{\sqrt{\tan x}}\ \mathrm{d}x \Rightarrow_{\substack{u=\tan x\\ \mathrm{d}u=\sec^2(x)\mathrm{d}x}} \int_0^1 \frac{1+u^2}{\sqrt{u}}\mathrm{d}u = \left[2\sqrt{u} + \frac{2}{5}u^{5/3}\right]_0^1 = \frac{12}{5}.$$

16. Common Exam III Solutions, Fall 2021

Question 1: Find the sum of the following series:

$$(a) \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)}, \qquad (b) \sum_{n=1}^{\infty} \frac{2^{2n}+4^n}{6^n}$$

Solution 1(a):

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} &= \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdot\cdot \\
&= \lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right] \\
&= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) \\
&= 1.
\end{aligned}$$

Solution 1(b):

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{2n}+4^n}{6^n} &= \sum_{n=1}^{\infty} \frac{4^n+4^n}{6^n} \\
&= 2\sum_{n=1}^{\infty} \frac{4^n}{6^n} \\
&= 2\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \\
&= \frac{2}{(1-\frac{2}{3})} \\
&= 6.
\end{aligned}$$

Note: for 1(b) we are using **geometric series** sum:

$$\sum_{n=1}^{\infty} r^n = \frac{1}{(1-r)}, |r|< 1.$$

Question 2(a): Use the **integral test** to determine whether the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

Solution 2(a): Let $f(x) = \frac{1}{x(\ln(x))^2}$.

(1) $f(x)$ is a continuous and non-negative function on $(2,\infty)$.
(2) $f(x)$ is a non-decreasing function on $(2,\infty)$, and

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,dx \rightarrow_{\mathrm{d}u=\frac{1}{x}\,dx}^{u=\ln(x)} \int \frac{1}{u^2}\,du$$
$$= -\frac{1}{u} = -\left[\frac{1}{\ln(x)}\right]_2^{\infty}$$
$$= \lim_{b\to\infty}\left(-\frac{1}{\ln(b)} + \frac{1}{\ln(2)}\right)$$
$$= \frac{1}{\ln(2)}.$$

Since the limit of the integral is finite, by the **integral test**, the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

converges.

Question 2(b): Use a **comparison test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{4}\right)^n$$

Solution 2(b) Let

$$b_n = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n.$$

Find $\lim_{n\to\infty} \frac{a_n}{b_n}$, where $a_n = \frac{1}{n}\left(\frac{1}{4}\right)^n$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Since, $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges (geometric series with $r = \frac{1}{4}$),

By the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

Question 3(a): Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\ 9^n}$$

Solution 3(a): Given,

$$a_n = \frac{(n+2)!}{n!\,9^n},$$
$$a_{n+1} = \frac{(n+3)!}{(n+1)!\ 9^{n+1}}$$

We have,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+3)!\ \ (n)!\ \ 9^n}{(n+2)!(n+1)!\ \ 9^{n+1}} \\
&= \frac{1}{9} \lim_{n\to\infty} \frac{(n+3)}{(n+1)} \\
&= \frac{1}{9} \lim_{n\to\infty} \frac{(1+\frac{3}{n})}{(1+\frac{1}{n})} \\
&= \frac{1}{9} < 1.
\end{aligned}$$

By **ratio test** the series

$$\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\ \ 9^n}$$

converges.

Question 3(b): Use the **root test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{3n}\right)^n$$

Solution 3(b): Here,

$$a_n = \left(\frac{1}{2} + \frac{1}{3n}\right)^n,$$

and

$$\begin{aligned}
\lim_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\left(\frac{1}{2} + \frac{1}{3n}\right)^n\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{3n}\right) \\
&= \frac{1}{2} < 1.
\end{aligned}$$

By **ratio test** the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{3n}\right)^n$$

converges.

Question 4(a): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{4n^3+n}}$$

.

Solution 4(a): Here $a_n = (-1)^n \frac{n}{\sqrt{4n^3+n}}$, $|a_n| = \frac{n}{\sqrt{4n^3+n}}$, and let $b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$.

$$
\begin{aligned}
\lim_{n\to\infty}\frac{|a_n|}{b_n} &= \lim_{n\to\infty}\frac{n\, n^{\frac{1}{2}}}{\sqrt{4n^3+n}}\\
&= \lim_{n\to\infty}\frac{n\, n^{\frac{1}{2}}}{n^{\frac{3}{2}}\sqrt{4+\frac{1}{n^2}}}\\
&= \lim_{n\to\infty}\frac{1}{\sqrt{4+\frac{1}{n^2}}}\\
&= \frac{1}{2} > 0.
\end{aligned}
$$

Since the series, $\sum_{n=1}^{\infty} b_n$ diverges (by p-test). By **comparison test**, the series $\sum_{n=1}^{\infty}|a_n|$ diverges. Therefore, the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{4n^3+n}}$$

is not absolutely convergent.
Using the alternating series test on the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{4n^3+n}} = \sum_{n=1}^{\infty}(-1)^n a_n,$$

we have $a_n = \frac{n}{\sqrt{4n^3+n}}$.

Observe,

(1) $a_{n+1} < a_n$, for all $n \geq 1$,
(2) $a_n > 0$, for all $n \geq 1$,, and
(3) $\lim_{n\to\infty} a_n = 0$.

Therefore, by **alternating series test** the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{4n^3+n}}$$

converges **conditionally.**

Question 4(b): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1}$$

Solution 4(b): Here $a_n = \sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1}$, $|a_n| = \sum_{n=1}^{\infty}\frac{e^n}{e^{2n}+1}$,

$$\text{Let } b_n = \sum_{n=1}^{\infty} e^n$$

$$\begin{aligned}\lim_{n\to\infty}\frac{|a_n|}{b_n} &= \lim_{n\to\infty}\frac{e^n}{e^n(1+e^{2n})}\\ &= \lim_{n\to\infty}\frac{1}{1+e^{2n}}\\ &= 1>0.\end{aligned}$$

The series, $\sum_{n=1}^{\infty} b_n$ diverges using by limit test. Since

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^n \neq 0$$

By **comparison test**, the series $\sum_{n=1}^{\infty}|a_n|$ diverges. Therefore, the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1}$$

is not absolutely convergent.

Using alternating series test on the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1} = \sum_{n=1}^{\infty}(-1)^n\ a_n$$

we have $a_n = \frac{e^n}{e^{2n}+1}$.

(1) $a_{n+1} < a_n$, for all $n \geq 1$,
(2) $a_n > 0$, for all $n \geq 1$, and
(3) $\lim_{n\to\infty} a_n = 0$.

Therefore, by **alternating series test** the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1}$$

converges **conditionally.**

Question 5(a): Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty}\left(\frac{2+n^2}{1+2n^2}\right)^n$$

Solution 5(a): Here

$$a_n = \left(\frac{2+n^2}{1+2n^2}\right)^n$$

$$\begin{aligned}\lim_{n\to\infty}\Big(a_n\Big)^{1/n} &= \lim_{n\to\infty}\frac{2+n^2}{1+2n^2}\\ &= \lim_{n\to\infty}\frac{(\frac{2}{n^2}+1)}{(\frac{1}{n^2}+2)}\\ &= \frac{1}{2}<1.\end{aligned}$$

So, by **root test** the series

$$\sum_{n=1}^{\infty}\left(\frac{2+n^2}{1+2n^2}\right)^n$$

converges.

Question 5(b): Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{1/n}$$

Solution 5(b): Here $a_n = \left(\frac{1}{2}\right)^{1/n}$. Find the $\lim_{n\to\infty} a_n$.

Let $y = a_n = \left(\frac{1}{2}\right)^{1/n} \Rightarrow \ln(y) = \ln\left(\frac{1}{2}\right)^{1/n} \Rightarrow \ln(y) = \frac{1}{n}\ln\left(\frac{1}{2}\right)$.

Taking limits both sides

$$\begin{aligned}\lim_{n\to\infty}\ln(y) &= \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{2}\right)\\ &= 0.\end{aligned}$$

Taking exponential both sides

$$\lim_{n\to\infty} e^{\ln(y)} = e^0 \Rightarrow \lim_{n\to\infty} y = \lim_{n\to\infty} a_n = 1 \neq 0$$

By **limit test**, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{1/n}$$

diverges.

Question 6(a) Write down the first 3 non-zero terms in the Maclaurin series for the function $f(x) = x + \cos(2x)$.

Solution 6(a) The Maclaurin series expansion is given by,

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots,$$

where,
$f(0) = 1,\ f'(0) = 1, f''(0) = -4, \cdots\cdots\cdots$

$$f(x) = 1 + x + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(-4) + \cdots$$

Therefore, the first 3 non-terms are

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2}.$$

Question 6(b): Find the first 3 non-zero terms in the Taylor series about $a = 1$ for the function $f(x) = 2 - x^2$.

Solution 6(b): The Taylor series expansion is about $a = 1$ given by,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots,$$

where,

$$f(1) = 1, f'(1) = -2, f''(1) = -2, f^{(n)} = 0(\forall\ n \geq 3).$$

$$f(x) = 1 - 2(x-1) - \frac{2\ (x-1)^2}{2!} = 1 - 2(x-1) - (x-1)^2.$$

The first, second and third non-zero terms are the coefficients of $(x-1)^0, (x-1)^1, (x-1)^2$ respectively. Therefore, the first 3 non-zero terms are

$$a_0 = 1, a_1 = -2, a_3 = -1.$$

Question 7: Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n3^n}$$

Solution 7: The radius of convergence, R, is given by

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{a_n}$$

For the given problem, $a_n = \frac{1}{n\ 3^n}$, we have:

$$(a_n)^{1/n} = \frac{1}{3n^{1/n}}$$

So,

$$\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{3} \lim_{n\to\infty} \frac{1}{n^{1/n}}$$

Let $y = n^{1/n}$. This implies:

$$\ln(y) = \ln\left(n^{1/n}\right) = \frac{1}{n}\ln(n)$$

Now, take the limits on both sides:

$$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{\ln(n)}{n} \quad \left(\text{using L'Hôpital's Rule, as it is of the form } \frac{\infty}{\infty}\right)$$

This simplifies to:

$$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Taking the exponential of both sides gives:

$$\lim_{n\to\infty} y = \lim_{n\to\infty} n^{1/n} = 1$$

Therefore, the limit is:

$$\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{3} = \frac{1}{R} \Rightarrow R = 3$$

Interval of Convergence: The series converges when $|x+2| < R \Rightarrow |x+2| < 3 \Rightarrow -5 < x < 1$. Now, we need to check the endpoints $x = -5$ and $x = 1$.

Case 1: When $x = -5$, the series becomes:

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-5+2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Using the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
Case 2: When $x = 1$, the series becomes:

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(1+2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Using the **p-test**, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Therefore, the **interval of convergence** is $-5 \leq x < 1$.
Question 8(a): Solve for x:

$$1 + x + x^2 + x^3 + \cdots = 2$$

Solution 8(a): Observe,

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = 2$$

Solving, we get $x = \frac{1}{2}$.
Question 8(b): Find the Taylor polynomial of order 2 generated by $f(x) = \ln(x)$ about $a = 1$.
Solution 8(b): The Taylor series expansion about $a = 1$ is given by:

$$f(x) = f(1) + (x-1) \cdot f'(1) + \frac{(x-1)^2}{2!} \cdot f''(1) + \frac{(x-1)^3}{3!} \cdot f'''(1) + \cdots,$$

where,

$$f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = -2, \ldots$$

$$f(x) = (x-1) - \frac{(x-1)^2}{2!} - \frac{2(x-1)^3}{3!} + \cdots$$

Therefore, the Taylor polynomial of order 2 is:

$$P_2(x) = (x-1) - \frac{(x-1)^2}{2!}.$$

17. Common Exam I Solutions, Spring 2022

Question 1: Find the length of the curve $y = \frac{1}{6}(2+4x^2)^{3/2}$ over $0 \leq x \leq 3$.

Solution 1: The length of the curve is given by:

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where:

$$\frac{dy}{dx} = 2x\sqrt{2+4x^2}$$

Now, let's calculate $\left(\frac{dy}{dx}\right)^2$:

$$\left(\frac{dy}{dx}\right)^2 = 4x^2(2+4x^2)$$

Therefore, $\left(1 + \left(\frac{dy}{dx}\right)^2\right) = (1+4x^2)^2$.
Now, we can evaluate the length of the curve:

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 (1+4x^2)\, dx = \left[x + \frac{4}{3}x^3\right]_0^3 = 39.$$

Question 2: Find the area of the surface formed by rotating the curve $y = \sqrt{6x - x^2}$ for $1 \leq x \leq 3$ about x-axis.
Solution 2: The area of the surface,

$$S = 2\pi \int_1^3 y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; \mathrm{d}x$$

$$\left(\frac{dy}{dx}\right) = \frac{1}{2\sqrt{6x-x^2}}(6-2x) \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(6-2x)^2}{4(6x-x^2)}$$

$$y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \;=\; 3$$

$$\boxed{S = 2\pi \int_1^3 y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; \mathrm{d}x = 12\pi}$$

Question 3: The base of a solid is the region bounded by the curves, $y = \sqrt{x}, y = 0$, and $x = 4$. The cross-sections perpendicular to the x-axis are squares. Find the volume of this solid.
Solution 3: Side of the square, $s = \sqrt{x}$; area of the square, $A(x) = (\sqrt{x})^2 = x$.
Volume of the solid,

$$V = \int_0^4 A(x)\; \mathrm{d}x = \int_0^4 x\; \mathrm{d}x = \; 8.$$

Question 4: A force of $F = \frac{x}{\sqrt{x^2+9}}$ lbs is applied to move an object along the x-axis from $x = 0$ to $x = 4$ft. Determine the amount of work done.
Solution 4: The work done is given by

$$W = \int_0^4 F\,dx = \int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx \rightarrow_{\substack{u=x^2+9 \\ \mathrm{d}u=2x\,dx}} = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} = \left[\sqrt{x^2+9}\right]_0^4 = (5-3) = 2.$$

Question 5: The region between the curve $y = e^x + e^{-x}$, $-1 \le x \le 1$, and the x-axis is revolved around the x-axis to generate a solid. Find its volume.
Solution 5: The volume for solid or revolution for a curve is the volume of object determined by a curve $f(x)$ rotated around the x-axis on an interval $[a, b]$ given by,

$$V = \int_a^b \pi\ (f(x))^2\ \mathrm{d}x$$

$$y = f(x) = e^x + e^{-x} \Rightarrow (f(x))^2 = e^{2x} + e^{-2x} + 2$$

The volume,

$$V = \int_{-1}^1 \pi(e^{2x} + e^{-2x} + 2)\ \mathrm{d}x = 4\pi + \pi(e^2 - e^{-2}).$$

Question 6(a): Find the derivative

$$y = x\ \ln\left(\cosh(2x)\right).$$

Solution 6(a): Using the chain rule,

$$\begin{aligned}
\frac{dy}{dx} &= x\frac{d}{dx}\ln\left(\cosh(2x)\right) + \ln\left(\cosh(2x)\right)\frac{d}{dx}x \\
&= x\frac{1}{\cosh(2x)}\frac{d}{dx}(\cosh(2x)) + \ln\left(\cosh(2x)\right) \\
&= x\tanh(2x)\frac{d}{dx}(2x) + \ln\left(\cosh(2x)\right) \\
&= 2x\tanh(2x) + \ln\left(\cosh(2x)\right).
\end{aligned}$$

Question (b): Evaluate the integral

$$\int_0^{\ln 2} \left(2e^x\ \cosh(x) - \mathrm{e}^{2x}\right)\ \mathrm{d}x$$

Solution 6(b): Simply $2e^x\ \cosh(x) = e^{2x} + 1$

$$\int_0^{\ln 2} \left(2e^x\ \cosh(x) - \mathrm{e}^{2x}\right)\ \mathrm{d}x = \int_0^{\ln 2} e^{2x} + 1\ -\ e^{2x}\ \mathrm{d}x = \int_0^{\ln 2} \mathrm{d}x = \ln(2).$$

Question 7. A 20 lb bucket is lifted from the ground into the air by pulling in L ft. The cable weighs 4 lb/ft. Of 400 ft-lbs of work done lifting both the bucket and cable, what is the length of the cable, L?

Solution 7: The work done to lift the bucket is denoted as W_1 and can be expressed as:

$$W_1 = 20L.$$

Similarly, the work done to lift the cable is represented as W_2 and can be calculated through integration:

$$W_2 = 4\int_0^L (L-y)\,dy = 4\left[Ly - \frac{y^2}{2}\right]_0^L = 4\left(L^2 - \frac{L^2}{2}\right) = 2L^2.$$

The total work done lifting both the bucket and cable, $W = 400 = W_1 + W_2$,

$$\Rightarrow 400 = 20L + 2L^2 \Rightarrow (L-10)(L+20) = 0 \Rightarrow L = 10.$$

Question 8. A tank is constructed by revolving the curve $y = 6x^2$ for $0 \leq x \leq 1$ about the y-axis. The tank is filled with fluid weighing 20 lb/ft^3. How much work is done in pumping all the fluid to a level 2 ft above the rim of the tank?

Solution 8: The work done in pumping all the fluid to a level 2 ft above the rim of the tank:

$$W = \int_0^6 \frac{20\pi}{6}\ y(8-y)\ \mathrm{d}y = \frac{20\pi}{6}\left[\frac{8y^2}{2} - \frac{y^3}{3}\right]_0^6 = 240\pi.$$

Question 9. The region enclosed by the curves $y = x+2, y = -x+2$ and $x = 3$ is revolved about the line $x = 6$ to generate a solid. Find the volume using the shell method.

Solution 9: The volume by shell method:

$$V = \int_a^b 2\pi r h\ \mathrm{d}x.$$

Here, $r = (x+2) - (-x+2) = 2x,\ h = (6-x)$ The volume,

$$V = \int_0^3 2\pi(2x)(6-x)\mathrm{d}x = 4\pi\left[\frac{6x^2}{2} - \frac{x^3}{3}\right]_0^3 = 72\pi.$$

Question 10. Find the volume of the solid of revolution formed by revolving the y-axis the region enclosed by

$$y = \cos\left(x^2\right)$$

and the $x-$axis.

Solution 10: Using the Shell method, we have,

$$V = \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x\cos\left(x^2\right)\mathrm{d}x$$
$$= \pi\left[\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}} = \pi.$$

18. Common Exam II Solutions, Spring 2024

1. (a) Find the slope of the tangent line to the curve $y = \arctan(\sinh(x))$ at $x = \ln(2)$.
b. Estimate the following integral using Simpson's rule with $n = 4$ steps.

$$\int_0^2 x^2 \sin^3(\pi x), \mathrm{d}x$$

2. Apply the Direct Comparison Test to determine if the following integral converges or diverges.

$$\int_\pi^\infty \frac{2+\cos x}{x} \quad \mathrm{d}x$$

b. Evaluate the following integral:

$$\int x(\ln x)^2 \quad \mathrm{d}x$$

3. (a) Evaluate the following integrals:

$$\int \frac{\mathrm{d}x}{(x^2-1)^{3/2}}$$

b.

$$\int \frac{x^2+2x+1}{(x^2+1)^2} \quad \mathrm{d}x$$

4. (a) Find the limit of the sequence if it exists:

$$a_n = \sqrt{n^2-n} - n$$

Sol:

$$\lim_{n\to\infty} a_n = (\sqrt{n^2-n} - n)\left(\frac{\sqrt{n^2-n}+n}{\sqrt{n^2-n}+n}\right) = \lim_{n\to\infty} \frac{-n}{\sqrt{n^2-n}+n} = -\frac{1}{2}.$$

b. Evaluate the following integral:

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x+1)}$$

5. (a) Evaluate the following integrals:

$$\int \tan(\theta)\Big(\cos^3(\theta) + \sec^3(\theta)\Big)\mathrm{d}\theta$$

Sol:

$$\begin{aligned}\int \tan(\theta)\Big(\cos^3(\theta) + \sec^3(\theta)\Big)\mathrm{d}\theta &= \int \sin(\theta)\cos^2(\theta) + \sin(\theta)\sec^4(\theta)\mathrm{d}\theta \\ &= -\frac{\cos^3(\theta)}{3} - \frac{1}{3\cos^3(\theta)} \\ &= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + \mathrm{C}.\end{aligned}$$

b.

$$\int x\sqrt{1-x} \quad \mathrm{d}x$$

Sol: Let $u = 1 - x \implies \mathrm{d}x = -\mathrm{d}u$.

$$\begin{aligned}\int x\sqrt{1-x} \quad \mathrm{d}x &= -\int \sqrt{u}(1-u)\mathrm{d}u \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \\ &= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + \mathrm{C}.\end{aligned}$$

6. (a) Apply the Limit Comparison Test to determine if the following integral converges:

$$\int_1^\infty \frac{\sqrt{e^{4x}+e^{-6x}}}{e^{3x}-1}\mathrm{d}x$$

b. Evaluate the following integral:

$$\int \frac{e^x + x}{xe^x}\mathrm{d}x$$

7. (a) Evaluate the following integrals:

$$\int_0^1 \frac{x^{1/3} - 2x^{2/3}}{x}\mathrm{d}x,$$

b.

$$\int \frac{\arcsin(2x)}{\sqrt{1-4x^2}}\mathrm{d}x$$