Problem Set 4

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1. Integrate

$$\int_0^\infty \frac{1}{1+x^2}\, \mathrm{d}x$$

2. Integrate

$$\int \frac{x^3}{\sqrt{1-x^2}}\, \mathrm{d}x$$

3. Integrate

$$\int \frac{1}{(4x^2+9)^2}\, \mathrm{d}x$$

4. Integrate

$$\int x \tan^{-1}(x)\, \mathrm{d}x$$

Solutions

1.

$$\int_0^\infty \frac{1}{1+x^2}\, \mathrm{d}x = \lim_{a\to\infty} \int_0^a \frac{1}{1+x^2}\, \mathrm{d}x = \lim_{a\to\infty} \Big[\tan^{-1}(x)\Big]_0^a = \lim_{a\to\infty} \tan^{-1}(a) = \frac{\pi}{2}.$$

2.

$$\int \frac{x^3}{\sqrt{1-x^2}}\, \mathrm{d}x \Rightarrow_{\substack{u=1-x^2 \\ \mathrm{d}u=-2x\ \mathrm{d}x}} -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\, \mathrm{d}u = -\frac{1}{2}(2\sqrt{u} - \frac{2}{3}u^{\frac{3}{2}}) = \frac{1}{3}(1-x^2)^{\frac{3}{2}} - \sqrt{1-x^2} + C$$

3.

$$\int \frac{1}{(4x^2+9)^2}\, \mathrm{d}x \Rightarrow_{\substack{x=\frac{3}{2}\tan(u) \\ \mathrm{d}x=\frac{3}{2}\sec^2(u)\mathrm{d}u}} \int \frac{\frac{3}{2}\sec^2(u)}{\left(9\tan^2(u))+9\right)^2}\mathrm{d}u = \frac{1}{54}\int \frac{\sec^2(u)}{\sec^4(u)}\mathrm{d}u = \frac{1}{54}\int \cos^2(u)\mathrm{d}u$$

$$= \frac{1}{54}\int \cos^2(u)\mathrm{d}u = \frac{1}{54}\int \frac{1+\cos(2u)}{2}\, \mathrm{d}u$$

$$= \frac{1}{108}\left(u + \frac{\sin(2u)}{2}\right) = \frac{1}{108}\left(\tan^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}\sin\left(2\tan^{-1}\left(\frac{2x}{3}\right)\right)\right) + C$$

4. Use integration by parts,

$$\begin{aligned}
\int x \tan^{-1}(x)\, \mathrm{d}x &= \tan^{-1}(x) \int x\, \mathrm{d}x - \int \frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}(x)) \int x\, \mathrm{d}x \\
&= \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\mathrm{d}x \\
&= \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}[x - \tan^{-1}(x)] \ + C
\end{aligned}$$

$$\int \frac{x^2}{1+x^2}\mathrm{d}x = \int \frac{(1+x^2)-1}{1+x^2}\mathrm{d}x = \int \left(1 - \frac{1}{1+x^2}\right) \mathrm{d}x = x - \tan^{-1}(x) \ + \ C$$